

True Rankings

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1 Introduction

Many professional soccer leagues have been halted before the intended number of matches was played. For some of these leagues it has been decided that the competition is over; for others it is still unclear whether or not matches will resume. For instance, Ligue 1 in France, and the Eredivisie in the Netherlands will not resume matches; for these leagues the season is over.

A relevant question now is to produce a final ranking. It is NOT a good idea to use the ranking that is based on the number of points collected sofar. Indeed, this suffers from the following arguments:

- Teams have played a different number of matches. This is actually the case in all major leagues (see below), and clearly is something that should not be ignored when making up a final ranking.
- Teams have played against different sets of opponents with varying strengths. When stopping the league prematurely, this is unavoidable; and this clearly causes a ranking based on the number of points collected sofar to be a biased one. For instance, two teams fighting against relegation whose set of remaining opponents differ hugely in strength should not be ranked judged on the number of collected points sofar.

Moreover, there is an elegant way to produce a ranking that accurately reflects the strengths of the teams. This method is known as the *direct ranking method*, and it takes into account both the number of matches played, and the strength of the opponents played against.

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2 The direct ranking method

We describe the direct ranking method; we refer to Keener [1] for a very readable description. It is interesting to see that already more than a 100 years ago, this method was used in a sports context, see Landau [2].

Assume there is a set of teams N . It is our goal to rank them while not all matches between each pair of teams may have been played. A basic assumption is that each team has a strength that is revealed when matches are played and outcomes are realized. We use the following symbols:

- r_i : strength of team $i, i \in N$.
- $a_{i,j}$: score that depends on the outcome of the matches (if any) played between two distinct teams i and $j, i, j \in N$.
- n_i : number of matches played by team $i, i \in N$.

To specify the parameters $a_{i,j}$, we follow the point distribution that is used in soccer: 0 points in case of a loss, 1 point in case of a tie, and 3 points for a win. So to be explicit: if teams i and j have played each other once, then $a_{i,j}$ equals 0 when team i lost to team j , $a_{i,j}$ equals 1 when team i drew against j , and $a_{i,j}$ equals 3 when team i beat team j . If teams i and j have played each other twice, then $a_{i,j}$ equals 0 when team i lost twice to team j , $a_{i,j}$ equals 1 when team i lost once, and drew once against team j , $a_{i,j}$ equals 2 when team i drew twice against team j , $a_{i,j}$ equals 3 when team i beat team j once, and lost to team j once, $a_{i,j}$ equals 4 when team i beat team j once, and drew once against j , and $a_{i,j}$ equals 6 when team i beat team j twice.

The total score of team $i \in N$ is denoted by s_i , and we express it as follows:

$$s_i = \frac{1}{n_i} \sum_{j=1}^{|N|} a_{i,j} r_j.$$

Notice that the score of each team is normalized by dividing the total weighted strength by the number of matches played.

As we assume that the total score is proportional to a team's strength, we expect this equality to hold: $s_i = \lambda r_i$ for some $\lambda > 0$.

In other words:

$$Ar = \lambda r,$$

where the elements of the matrix A equal $\frac{a_{i,j}}{n_i}$, $i, j \in N$. It follows that the strengths of the teams is captured by an eigenvector of A . The Perron-

Frobenius theorem describes exactly under which conditions on A this eigenvector has positive elements and is unique, see eg [4] for an explanation.

3 The results

We have done the computations for the following six leagues: Ligue 1 (Section 3.1), the Eredivisie (Section 3.2), the Premier League (Section 3.3), Serie A (Section 3.4), Bundesliga (Section 3.5) and the Primera Division (Section 3.6). More concretely, for each of these leagues, we have computed a (normalized) eigenvector of the matrix A as defined in Section 2, thereby revealing the strengths of the teams based on nothing else but the results realized sofar. Thus, the rankings we display in the next subsections are the rankings that follow from the strengths we computed, and if the ranking of a team differs from the ranking by number of collected points, we add between brackets the current ranking after the team's name.

For each of these leagues, we shortly comment on the current state, and possible repercussions. Notice that for Ligue 1 and the Eredivisie, the season is indeed over. An up-to-date site on the likelihoods of resuming the European soccer leagues can be found at Sports Illustrated (SI) [3].

3.1 Ligue 1: the true ranking

All teams have played 28 of the 38 matches except PSG and Strasbourg which have played 27. Table 1 gives the ranking by strengths.

Rank	Team	Strength
1	PSG	0.403
2	Olympique Marseille	0.304
3	Lille (4)	0.272
4	Rennes (3)	0.270
5	Reims	0.261
6	AS Monaco (9)	0.229
7	Nice (6)	0.220
8	FC Nantes (13)	0.211
9	Angers (10)	0.207
10	Montpellier (8)	0.204
11	Strasbourg	0.201
12	Olympique Lyonnais (7)	0.199
13	Bordeaux (12)	0.197
14	Dijon (16)	0.194
15	Metz	0.189
16	Brest (14)	0.180
17	Saint-Étienne	0.163
18	Nîmes	0.153
19	Amiens	0.148
20	Toulouse	0.072

Figure 1: The true ranking in Ligue 1

The Ligue de Football Professionnel (LFP) has decided that the season is over. In addition, on April 30, the LFP decided upon the following:

- 1 PSG becomes champion and enters the Champions League,
- 2 Toulouse and Amiens relegate,
- 3 Olympique Marseilles and Stade Rennes also go the Champions League, and
- 4 Lille, Stade de Reims and Nice go to the Europa League.

While decisions 1 and 2 are consistent with the ranking computed above, decisions 3 and 4 are not. Indeed, based on the existing results, it is Lille

that should receive a CL ticket instead of Stade Rennes. In addition, Stade Rennes should receive a Europa League ticket, just as AS Monaco, and Reims (and not Nice).

3.2 Eredivisie: the true ranking

All teams have played 26 of the 34 matches except Ajax, AZ, Feyenoord, and FC Utrecht, which have played 25. Table 2 gives the ranking by strengths.

Rank	Team	Strength
1	AZ (2)	0.361
2	Ajax (1)	0.342
3	Feyenoord	0.324
4	PSV	0.292
5	Willem II	0.290
6	FC Utrecht	0.258
7	Vitesse	0.256
8	FC Groningen (9)	0.236
9	Heracles (8)	0.216
10	SC Heerenveen	0.204
11	FC Emmen (12)	0.198
12	Sparta (11)	0.186
13	FC Twente (14)	0.181
14	VVV (13)	0.170
15	Fortuna Sittard (16)	0.155
16	PEC Zwolle (15)	0.144
17	ADO Den Haag	0.107
18	RKC	0.101

Figure 2: The true ranking in the Eredivisie

The Dutch soccer association (KNVB) has decided upon the following: no champion is determined, no teams will relegate, and no teams from the second division will be promoted. However, the KNVB also decided that Ajax receives a ticket for CL, while, according to the results in Figure 2, that ticket should go to AZ.

3.3 Premier League: the true ranking

All teams have played 29 of the 38 matches except Manchester City, Sheffield United, Arsenal, and Aston Villa who have played 28. The hope is that play can resume at June 8 (see SI [3]). Our results can be found in Figure 3.

Rank	Team	Strength
1	Liverpool	0.451
2	Manchester City	0.309
3	Manchester United (5)	0.283
4	Leicester (3)	0.263
5	Chelsea (4)	0.251
6	Wolverhampton	0.231
7	Arsenal (9)	0.215
8	Sheffield United (7)	0.213
9	Tottenham (8)	0.205
10	Burnley	0.200
11	Everton (12)	0.198
12	Newcastle United (13)	0.194
13	Crystal Palace (11)	0.189
14	Southampton	0.177
15	Watford (17)	0.162
16	Brighton (15)	0.146
17	Bournemouth (18)	0.141
18	West Ham United (16)	0.138
19	Norwich (20)	0.127
20	Aston Villa (19)	0.123

Figure 3: The true ranking in the Premier League

While awarding the title to Liverpool based on these results will meet little discussion, deciding upon the top 4 is another matter. Our results find that next to Liverpool, it is Manchester United, Manchester City, and Leicester that should qualify (and not Chelsea).

When it comes to relegation, the situation might lead to even more debate. According to Figure 3, the three weakest teams are, next to Norwich and Aston Villa, West Ham United (which is currently ranked 16th).

3.4 Serie A: the true ranking

All teams have played 26 of the 38 matches except Inter, Atalanta, Verona, Parma, Sassuolo, Cagliari, Torino, and Sampdoria who have played 25. According to SI [3], matches are scheduled to resume behind closed doors in the beginning of June.

Our results can be found in Figure 4.

Rank	Team	Strength
1	Juventus	0.385
2	Lazio	0.378
3	Inter	0.330
4	Atalanta	0.290
5	AS Roma	0.258
6	Napoli	0.219
7	Hellas Verona (8)	0.210
8	Parma (9)	0.206
9	Cagliari (12)	0.202
10	Bologna	0.191
11	AC Milan (7)	0.186
12	Sassuolo (11)	0.176
13	Fiorentina	0.174
14	Torino (15)	0.171
15	Udinese (14)	0.156
16	Lecce (18)	0.146
17	Genoa	0.142
18	Sampdoria (16)	0.142
19	SPAL	0.136
20	Brescia	0.085

Figure 4: The true ranking in the Serie A

It has been decided that, if play is not resumed, the last three in the current ranking will be relegated. According to the results in Figure 4, this is especially hard (and unjust) for Lecce, which, when it comes to strength, is 16th out of 20.

3.5 Bundesliga: the true ranking

It is unclear whether the Bundesliga will resume - a decision is expected Wednesday May 6.

Rank	Team	Strength
1	Bayern München	0.342
2	Borussia Dortmund	0.340
3	RB Leipzig	0.326
4	Borussia Mönchengladbach	0.307
5	Bayer Leverkusen	0.302
6	1899 Hoffenheim (9)	0.247
7	Schalke 04 (6)	0.238
8	SC Freiburg	0.235
9	VfL Wolfsburg (7)	0.229
10	Eintracht Frankfurt (12)	0.223
11	1. FC Köln (10)	0.203
12	Union Berlin (11)	0.200
13	Hertha BSC	0.175
14	FC Augsburg	0.157
15	FSV Mainz 05	0.142
16	Fortuna Düsseldorf	0.135
17	Werder Bremen	0.121
18	Paderborn	0.102

Figure 5: The true ranking in the Bundesliga

Apparently, in the Bundesliga, for the top ranked teams, as well as for the bottom ranked teams, the ranking by strengths is aligned with the ranking by number of points collected so far.

3.6 Primera Division: the true ranking

According to SI [3], it is not expected that matches resume before summer, if at all.

Rank	Team	Strength
1	Barcelona	0.352
2	Real Madrid	0.350
3	Sevilla	0.269
4	Athletico Madrid (6)	0.258
5	Valencia (7)	0.255
6	Real Sociedad (4)	0.254
7	Getafe (5)	0.251
8	Athletic de Bilbao (10)	0.224
9	Villareal (8)	0.223
10	Real Betis (12)	0.203
11	Levante (13)	0.203
12	Granada (9)	0.202
13	Osasuna (11)	0.189
14	Alaves	0.174
15	Celta de Vigo (17)	0.166
16	Eibar	0.164
17	Real Valladolid (15)	0.153
18	Real Mallorca	0.151
19	Leganes	0.131
20	Espanyol	0.121

Figure 6: The true ranking in the Primera Division

References

- [1] Keener, J. (1993), *The Perron-Frobenius theorem and the ranking of football teams*, SIAM Review **35**, 80–93.
- [2] Landau, E. (1915), *Über Preisverteilung bei Spielturnieren*, Zeitschrift für Mathematik und Physik **63**, 192–202.
- [3] Sports Illustrated, *The Ultimate Guide to When (or If) European Leagues Will Resume*, <https://www.si.com/soccer/2020/04/28/europe-leagues-uefa-plan-return-dates-coronavirus-matches> accessed May 1, 2020.
- [4] Wikipedia, *Perron Frobenius theorem*, https://en.wikipedia.org/wiki/Perron-Frobenius_theorem accessed May 1, 2020.