Optimization in Finance - Practice and Challenges

Darling, did you shrink my alpha?

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In this talk, we address:

1. **How do practitioners use optimization in finance?**
   - Introduction & Terms (Darling, look – this is my *alpha* !)
   - Modeling risk
   - Investment strategies
2. **Which practical needs are tackled with which techniques?**
   - The clue of Markowitz - Mean-Variance Optimization (and variants)
   - Exploiting estimation errors - Robust Optimization
   - Conquering dimension time - Multi-Period Optimization
3. **Which problems are still out there for optimization research?**
   - Round-lotting etc - General mixed-integer problems
   - Estimation errors in risk models – Semi-definite programming
   - Trade Scheduling, High Frequency Trading – Online optimization

n. **So...who did really shrink the alpha?**
   - Economy
   - Politics
   - Corona
   - George Dantzig
   - Ben-Tal & Nemirovski
   - Black & Litterman
   - Reality
   - The optimizer
   - Darling

Curriculum Vitae

Dr. Adrian Zymolka

- Math diploma at Philipps University Marburg
- Researcher at Zuse Institute Berlin (ZIB)
- PhD in Mathematical Optimization at TU Berlin
  (supervised by Prof. Dr. Dr. h.c. mult. Martin Grötschel)
- Optimization Consultant at atesio GmbH
- Joined Axioma in 2007
  - 2007: London - Director, Client Services Europe
  - 2010: New York - Senior Director, Client Services
  - 2015: Frankfurt - Managing Director, Axioma Deutschland
  - 2019: Qontigo = Axioma + STOXX + DAX

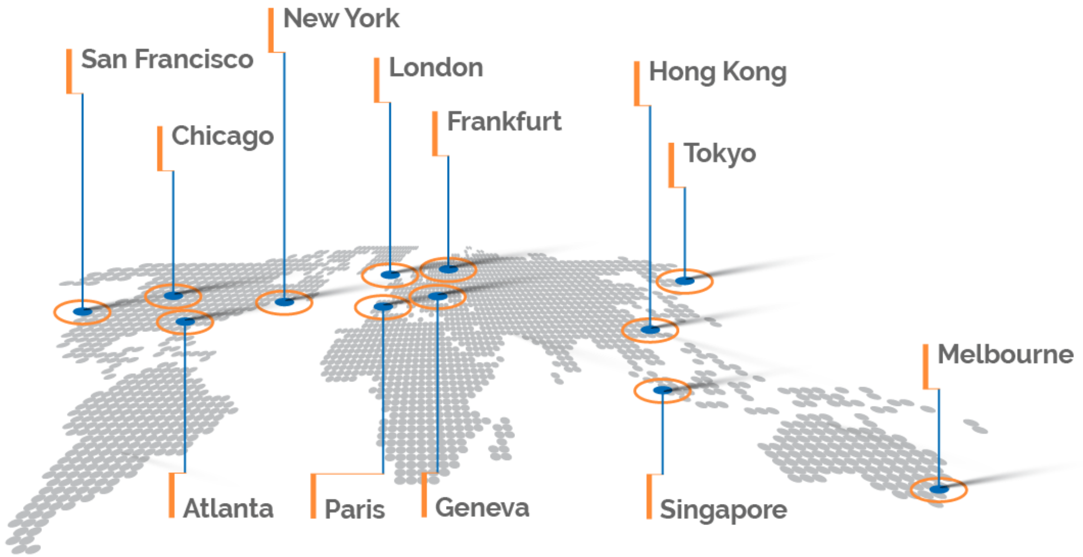
Qontigo (Analytics) in a Nutshell

350+ CUSTOMERS
GROWTH AT 25% CAGR

$2+ TRILLION
AUM DAILY OPTIMIZED

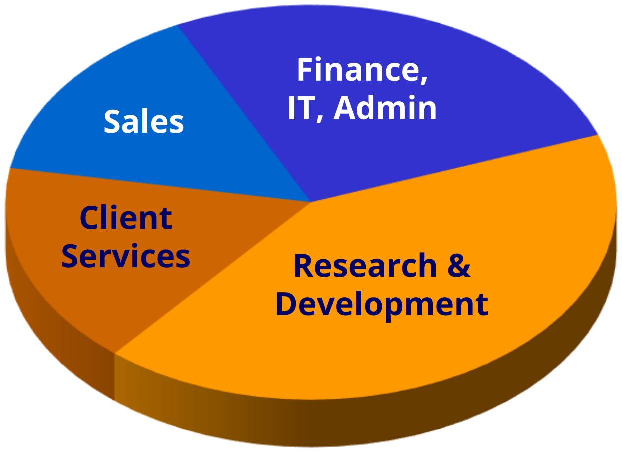
7000+
PROFESSIONAL USERS

75+ DAILY RISK MODELS
5000+ ANALYTIC FACTORS



Qontigo's Mission:

A financial intelligence powerhouse that delivers genuine competitive advantage to its clients - providing best-of-breed solutions powered by the latest technology, driven by continuous innovation and a strong entrepreneurial spirit.



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Introduction to Quantitative Finance

- **There are many application fields for math in finance:**
  - Credit business (loans, mortgages, etc) – stochastic models
  - Pricing of (complex) instruments – stochastic/numeric models
  - Exchanges & Trading – Market making, brokering ('auctions')
  - Mergers & Acquisitions (corporate actions) – operational, strategical
  - Interbank business, monetary policy – economic models

  Optimization, in some form or fashion, is used in most (if not all) of these areas.

- **We focus on an area which uses optimization in a methodical fashion:**
  portfolio management **(for equity portfolios)**
  - For a given set of assets, determine whether/how much to invest in each
  - Combination of a) asset selection, b) investment sizing
  - General distinction: fundamental vs. quantitative portfolio management
  - Many important characteristics to consider:
    concentration, leverage, "risk", return, transaction costs, exposures, , ...

  Let's clarify some more terms first….

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Alpha….and how can it shrink?

- **What is “alpha”?**

Some ideas….

…and potential alpha shrinkage…

TIER

Alpha Communication

alpha COMPANY
We can provide anything. From anywhere.

ALFA ROMEO

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Portfolio Management Terms and Concepts

- **Key terms in portfolio construction:**
  - **Alpha** = (expected) excess return of an investment asset
  - Beta = market sensitivity of an asset
  - Portfolio = investments in financial assets (e.g. equities)
  - Benchmark = public portfolio/index representing a market
  - Holding/weight = position in an asset / proportion in the portfolio
  - Trade = change in a position from one portfolio state to the next
  - Market impact = price shortfall for large trades
  - Risk = what you need to take to get above the *risk-free return (e.g. Libor)*…

- **Key concepts in portfolio management:**
  - Shorting = ‘negative’ position in an asset (sell a ‘borrowed’ assets)
  - Portfolio strategy = set of rules & goals to manage a portfolio
  - Backtest = historical simulation of a portfolio strategy
  - Diversification = exploitation of correlation effects by combining holdings
  - Hedging = control/reduction of certain risk types in a portfolio

- **“Risk” is a critical term** for portfolio managers….so let’s look closer…

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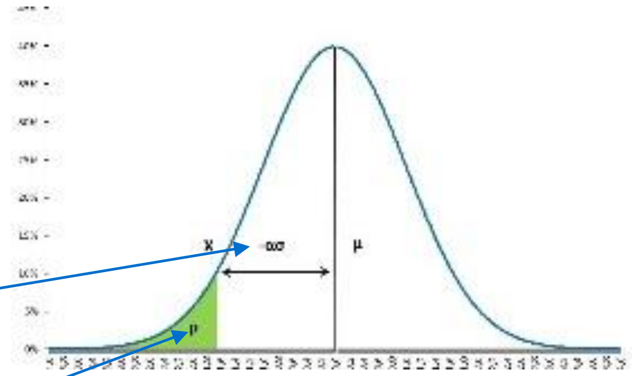
Risk Modeling in Finance

- **What is risk?**
  - Potential losses?
  - Probability of potential losses?
  - Can making too much revenue / return be risky, too?

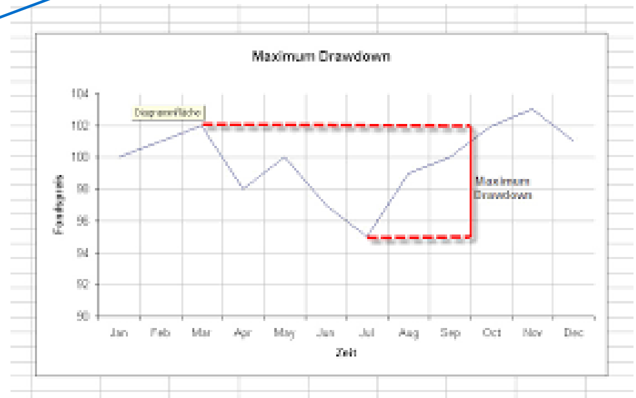
Most general: risk is the chance/probability to miss an expectation



- In finance, different risk measures are used:

  - Volatility: symmetric risk measure based on (co-) variances

  - Loss risk measures (one-sided risk, tail risk, percentiles):
    - Value at Risk (VaR) – tail risk and variants: CVaR, WCVaR, …
    - Drawdown (historic highest value drop)
    - Asymmetric risk measures (e.g., options, derivatives)



In portfolio construction, (so far) we stick with simplicity: risk = volatility measured in standard deviation or variance – using risk models

Risk Model Basics

- **Risk model** = asset-to-asset covariance matrix of asset return histories

$$Q = \begin{array}{cc} \begin{pmatrix} \sigma_{1,1} & \sigma_{1,2} & \cdots & \sigma_{1,N} \\ \sigma_{2,1} & \sigma_{2,2} & \cdots & \sigma_{2,N} \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_{N,1} & \sigma_{N,2} & \cdots & \sigma_{N,N} \end{pmatrix} & \begin{array}{c} 1 \\ 2 \\ \vdots \\ N \end{array} \ \text{assets} \\ \begin{array}{cccc} 1 & \quad 2 & \quad \dots & \quad N \end{array} & \\ \text{assets} & \end{array}$$

where $\sigma_{a,b}$ = covariance between assets a and b (variance if a=b)

Note: such matrices are positive semi-definite

- Portfolio risk : given the portfolio weights $w = (w_1, w_2, \dots, w_N)$,

  the portfolio risk is $w^T Q w$ (in *variance* terms)

  $\sqrt{w^T Q w}$ (in *std.dev.* terms)

Risk Model Types

- Risk model types:
  - *Dense models*: full asset-asset covariance matrix - not practical for >> 1,000 assets
  - *Factor models*: dimensional reduction approach

- Factor model: small set of "factors" representing key economic measures,

  reduce to a factor-factor covariance matrix Ω and

  link to assets via an asset-factor exposure matrix B

  plus asset specific risks δ_i in a diagonal matrix $\Delta = \mathrm{diag}(\delta_{1,\ \ldots,}\ \delta_N)$

$$\Omega = \begin{array}{cc} \begin{pmatrix} \sigma_{1,1} & \sigma_{1,2} & \cdots & \sigma_{1,F} \\ \sigma_{2,1} & \sigma_{2,2} & \cdots & \sigma_{2,F} \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_{F,1} & \sigma_{F,2} & \cdots & \sigma_{F,F} \end{pmatrix} & \begin{array}{c} 1 \\ 2 \\ \vdots \\ F \end{array} \text{factors} \\ \begin{array}{cccc} 1 & 2 & \cdots & F \end{array} & \\ \text{factors} & \end{array}$$

$$B = \begin{array}{cc} \begin{pmatrix} f_{1,1} & f_{1,2} & \cdots & f_{1,F} \\ f_{2,1} & f_{2,2} & \cdots & f_{2,F} \\ \vdots & \vdots & \vdots & \vdots \\ f_{N,1} & f_{N,2} & \cdots & f_{N,F} \end{pmatrix} & \begin{array}{c} 1 \\ 2 \\ \vdots \\ N \end{array} \text{assets} \\ \begin{array}{cccc} 1 & 2 & \cdots & F \end{array} & \\ \text{factors} & \end{array}$$

then the final risk model becomes $Q = \underbrace{B^T \Omega B}_{\text{factor risk = systematic (regressed)}} + \underbrace{\Delta}_{\text{idiosyncratic = specific residual risk}}$

and portfolio risk is calculated as $w^T Q w = w^T B^T \Omega B w + w^T \Delta w$

Factor Risk Model Types

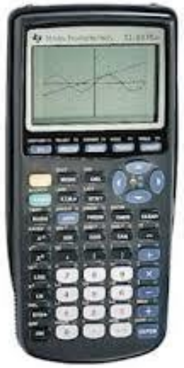
- Factor risk model types:
  - *Fundamental factors*: factors have defined financial meaning
  - *Statistical factors*: factors derived from statistical PCA methods
- Fundamental factors: typical financial measures with economic meaning in use:
  - style factors (size, momentum, volatility, value, growth,...)
  - market factor (intercept, market as a whole)
  - classification factors (industry, country, currency)

  with cross-sectional factor (co-)variances and exposures normalized (styles) or 0/1 (classifications, market);

  factor selections vary for different models

- Statistical factors: statistical risk model factors:
  - are derived by PCA methods directly from asset returns
  - have no obvious financial meaning
  - cover a maximum portion of the relevant risk space

  factor numbers vary for different models



- Specific risks: capture the residuals in both approaches (non-systematic risk)

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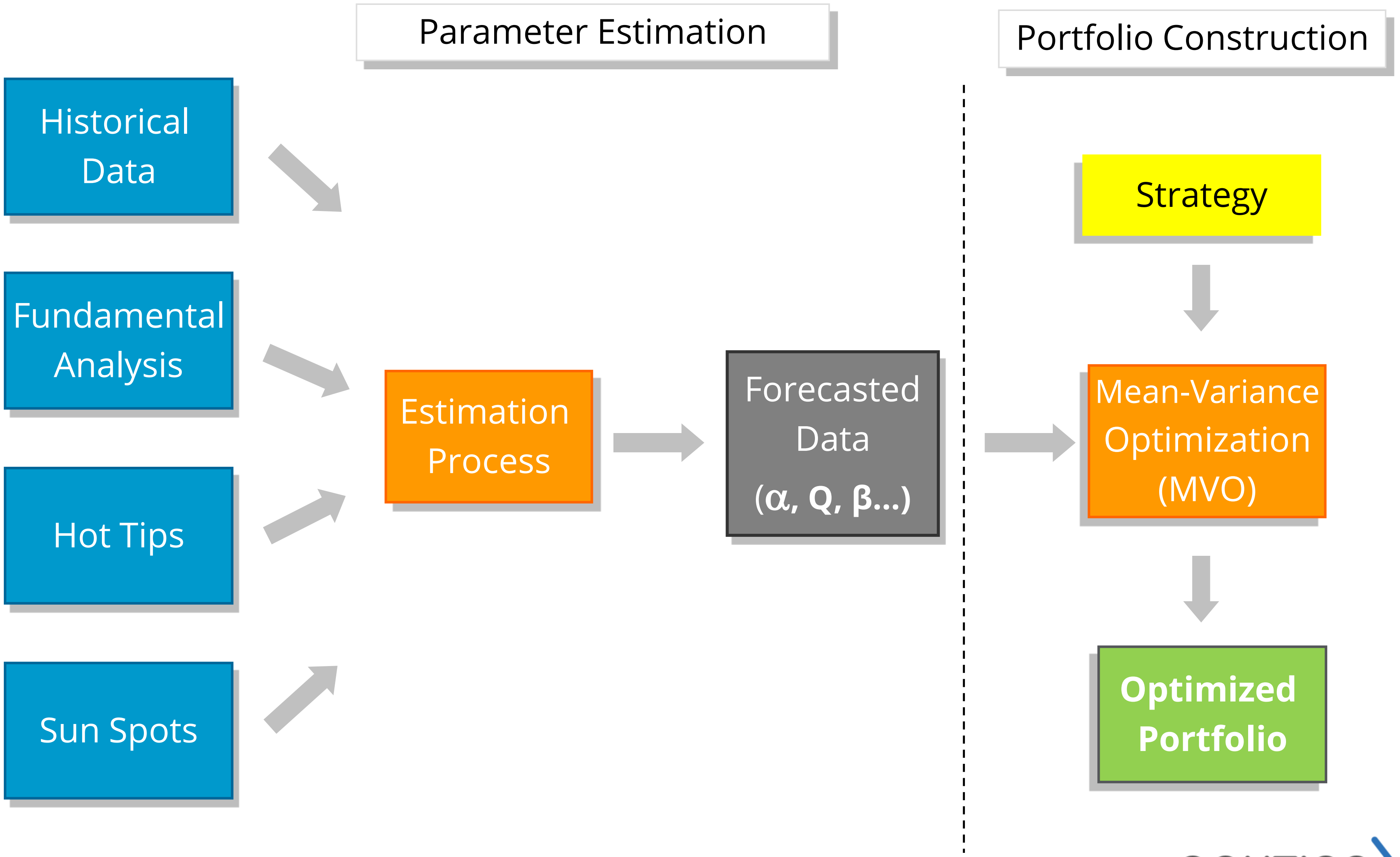
Investment Strategies

- **Strategy concepts used in practical portfolio management:**
  - Active vs. Passive Portfolios
  - Long-only vs. long-short vs. extension strategies (e.g., 130-30)
  - Market neutral, dollar neutral, risk strategy
  - Factor investing (Smart Beta)
  - Risk parity

- **Strategy components used in practical portfolio management:**
  - Position limits
  - Exposure limits (industries, countries, currencies, factors, …)
  - Beta limits (often around 0 = market neutral, or 1 = market match)
  - Turnover limits – or tcost minimization
  - Market impact inclusion
  - Count limits (number of names, number of trades)
  - Regulatory conditions (e.g. UCITS compliance)

Portfolio Management Process



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Portfolio Construction: Basic Models

- **Markowitz** (1950's) developed the first portfolio optimization models
- **Inputs**:

U - investment universe
α - expected returns (alpha)
Q - covariances of returns
λ - risk aversion parameter
B - budget
R - max limit on risk (std dev)
r - min bound on return

Minimize risk (variance)
Constraint on return

$$\begin{aligned} &\text{Min} \quad x^t Q x \\ &\text{st.} \quad \sum_{i \in U} x_i = B \\ &\qquad \alpha^t x \geq r \\ &\qquad x_i \geq 0, i \in U \end{aligned}$$

Quadratic Programming Problem

(linear constraints)

Maximize return
Constraint on risk (std)

$$\begin{aligned} &\text{Max} \quad \alpha^t x \\ &\text{st.} \quad \sum_{i \in U} x_i = B \\ &\qquad x^t Q x \leq R^2 \\ &\qquad x_i \geq 0, i \in U \end{aligned}$$

Quadratic Programming Problem

(quadratic constraints)

Mean-Variance Optimization (MVO)

$$\begin{aligned} &\text{Max} \quad \alpha^t x - \lambda \cdot x^t Q x \\ &\text{st.} \quad \sum_{i \in U} x_i = B \\ &\qquad x_i \geq 0, i \in U \end{aligned}$$

Quadratic Programming Problem

(linear constraints)

- **Outputs**: x - portfolio positions

Efficient Portfolios

Efficient Frontier: best (= optimal) trade-offs between risk and return (no portfolio can be above!)

Efficient Frontier

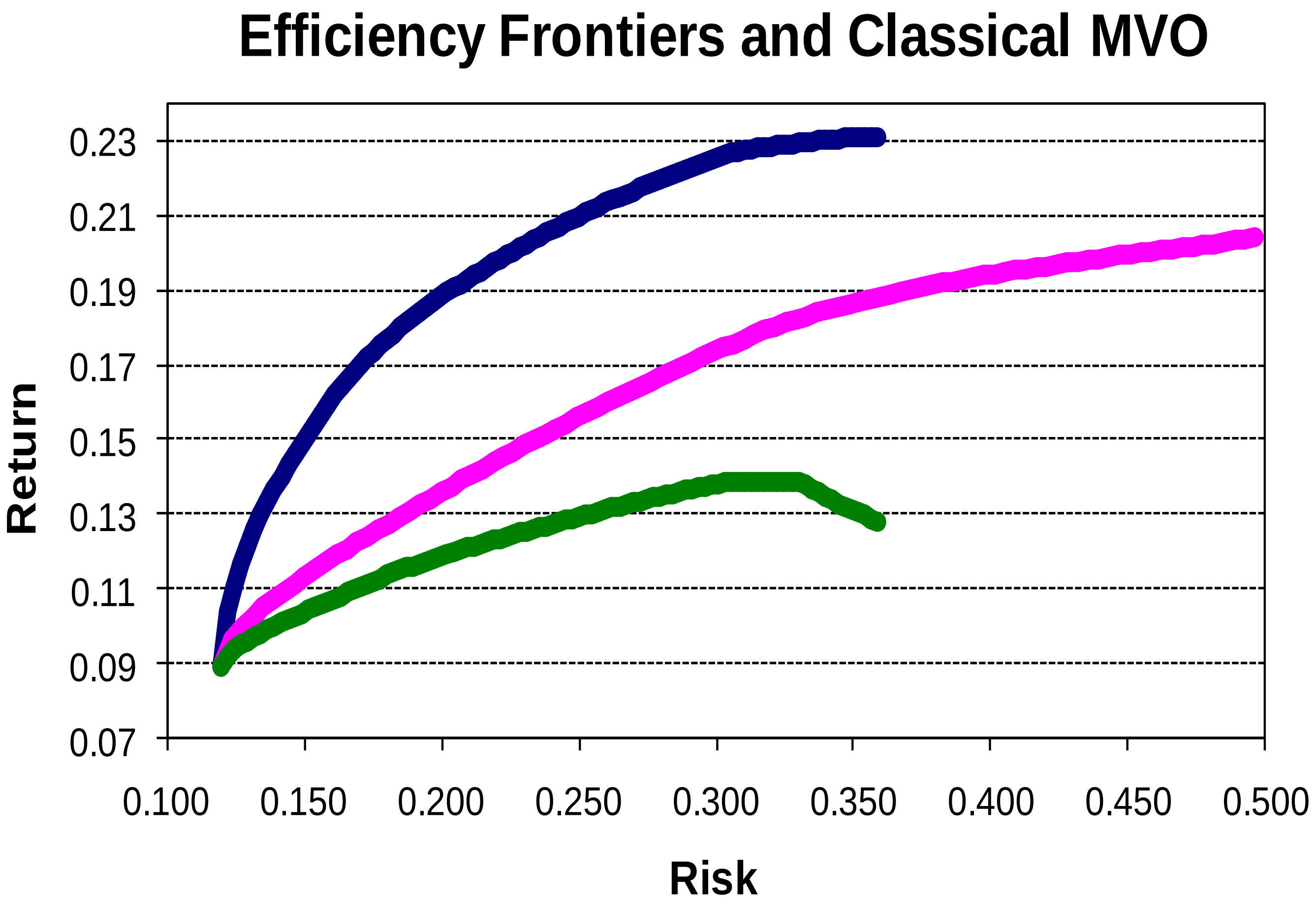
Efficiency Frontier computed using the estimated expected returns

True Frontier

Efficiency Frontier computed using the true future returns

Actual Frontier

Return for the portfolios in the Estimated Frontier using the true future returns



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Challenges in Quantitative Portfolio Construction

Part I: Methodological Issues

- **Shortcomings of Mean-Variance Optimization (MVO):**
  - Does my portfolio maximize alpha or errors?
  - Is mean-variance optimization choosing the “right” assets?
  - Are my portfolios “intuitive”?
  - Are my portfolios stable with respect to changes in the inputs?
  - Am I overestimating/underestimating risk and return?

- **Transferring information**
  - Is the (alpha) “information” properly translated to my portfolio?
  - How does my “strategy” affect this translation?
  - Are compliance restriction negatively affecting the portfolio’s performance (=realized alpha)?

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Challenges in Quantitative Portfolio Construction

Part II: Implementation Issues

- **Trading considerations**
  - How does trading affect my performance?
  - Can I incorporate trading considerations in the portfolio construction process?
  - What is a realistic market impact model?
  - Can I schedule my trades automatically?

- **Operational considerations**
  - How do I make sure that my portfolio is “implementable”?
  - How do I effectively “budget” risk?
  - How do I control that there are not too many small positions?

⇒ **require various basic MVO model extensions**

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Model Extensions – Linear/Quadratic Rules

- **Simple (Linear):**
  - Initial holdings (h)
  - Transaction variables ($t = |x - h|$)
  - Limits on holdings/trades ($x \leq u$, $t \leq v$)
  - Limits on industry/sector holdings ($\sum_{i \in S} x_i \leq c$)
  - Active holdings, industry/sector holdings ($|x - b| \leq u$, $\sum_{i \in S} |x_i - b_i| \leq c$)
  - Limits on turnover, trading, buys/sells ($\sum_{i \in S} t_i \leq c$)

- **Complex (Linear-Quadratic):**
  - Long/short portfolios (eliminate $x \geq 0$)
  - Multiple risk constraints ($x^t Q\, x \leq c$)
  - Multiple active risk/index tracking constraints ($(x - b)^t Q\,(x - b)) \leq c$)
  - Marginal contribution to risk constraints
  - Soft constraints/objectives
    - Add “slack” $s \geq 0$ so that $\sum_{i \in S} x_i \leq c$ is modified to $\sum_{i \in S} x_i - s \leq c$
    - Add s or s^2 to the objective as a penalty term

Model Extensions – Costs and Taxes

- **Transaction cost models:**
  - Linear (add to the objective $\sum_{i \in U} \gamma_i t_i$)
  - Piecewise Linear convex
  - Fixed Charge costs: if $t_i > 0$ then *add* c_i to the *objective*

- **Market impact models:**
  - Quadratic (add to the objective $\sum_{i \in U} \gamma_i t_i^2$)
  - Fractional powers (3/2, 5/3)
  - Piecewise Quadratic/Piecewise Linear
  - Combinations

- **Tax rules:**
  - Tax rates, wash sale rule
  - Tax liabilities
  - Long- and short-term gains/losses

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Model Extensions – Combinatorial

- **Thresholds:**
  - Holdings ($x_i = 0$ *or* $x_i \geq c$)
  - Transactions ($t_i = 0$ *or* $t_i \geq c$ ...*or* $x_i = 0$)
- **Cardinality limits:**
  - Holdings ($|\{i \mid x_i \neq 0\}| \leq c$)
  - Transactions ($|\{i \mid t_i \neq 0\}| \leq c$)
- **Round Lots:**
  - Transactions ($t_i \in \{k \cdot c \mid k = 1,2,\ldots\}$), c = lot size
- **If … then … else conditions:**
  - if {*linear expression*} then {*linear expression*} else {*linear expression*}
  - E.g.: one-sided constraints (limits only on longs / shorts / buys / sells)
- **Combinations:**
  - e.g., at most 20 stocks with holdings > 2%
  - UCITS 5/10/40 concentration avoidance:
    - No issuer more than 10% of portfolio
    - All issuers with more than 5% in sum to be less than 40%

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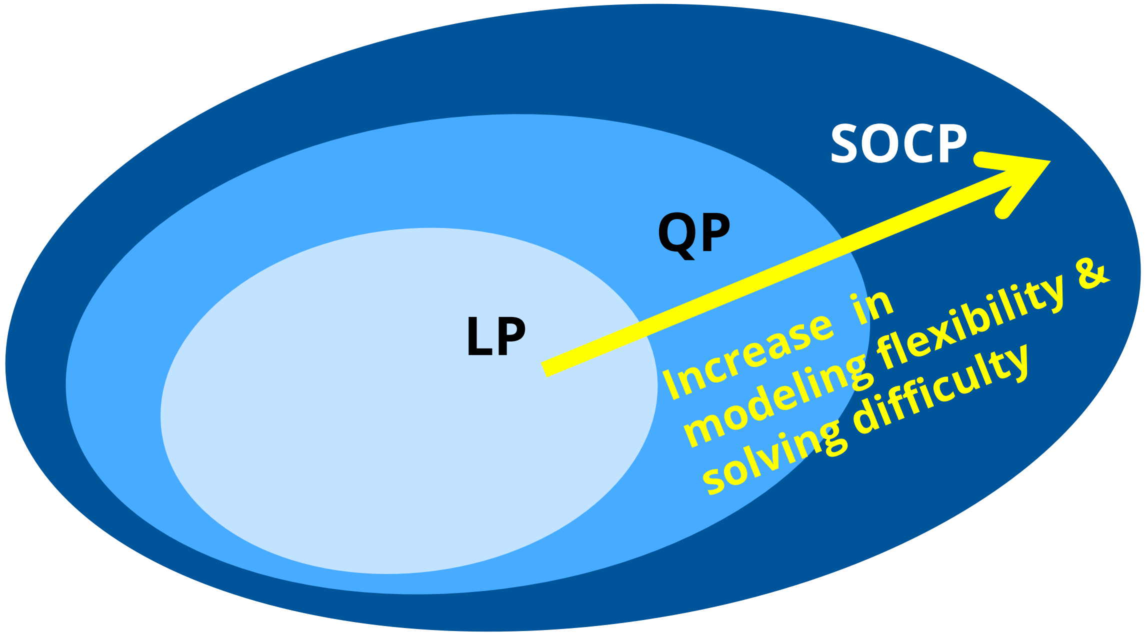
Second Order Cone Programming (SOCP)

Current optimization approaches in finance (core solver):



		Function Type		
Optimization Problem Class		Linear	Quad	SOC
LP	Objective	✓	✗	✗
	Constraints	✓	✗	✗
QP	Objective	✓	✓	✗
	Constraints	✓	✗	✗
SOCP	Objective	✓	✓	✓
	Constraints	✓	✓	✓

LP = Linear Programming
QP = Quadratic Programming
SOCP = Second-Order Cone Programming

State-of-the-art solving methodologies:

QP - 1950's (Markowitz)

SOCP - today (Axioma Portfolio Optimizer)

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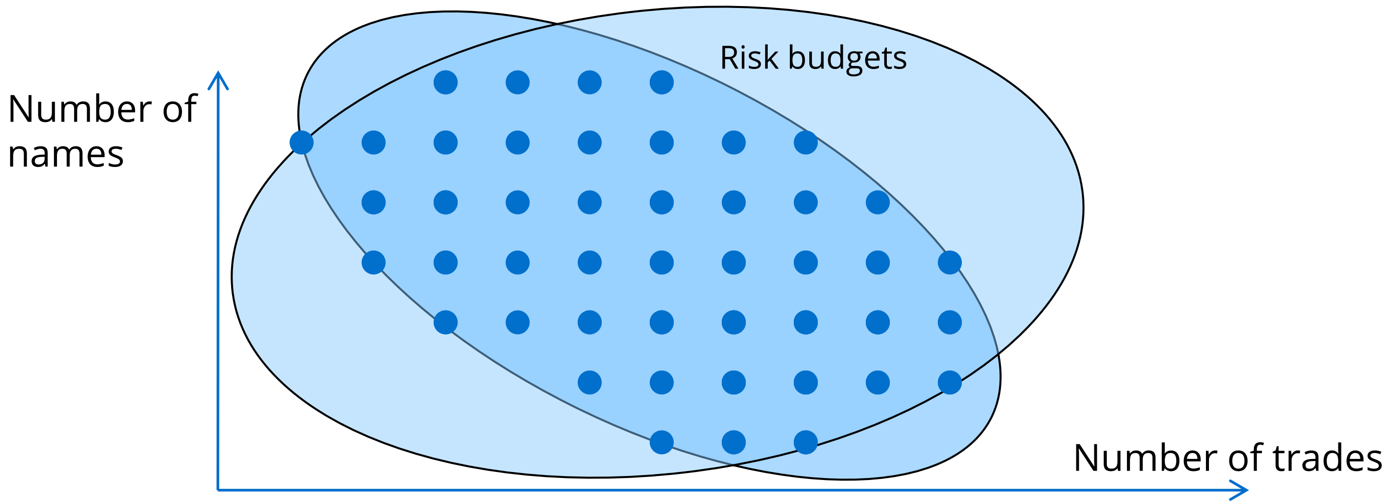
SOCP-IP adds Discrete Overlay

Axioma Portfolio Optimizer uniquely combines powerful core SOCP with a smart discrete method overlay:



5/10/40 rule

$$x_i \leq 10\% \quad \forall i \in U$$

$$\sum_{x_i \geq 5\%} x_i \leq 40\%$$

This methodology − tuned for portfolio construction tasks − enables:

Discrete decisions:

- Limit numbers of names or trades
- Thresholds on holdings and trades
- Fixed charge costs

Conditional limitations:

- Limits on long or short positions only
- Limits on buys or sells only
- Tax constraints (holdings and transaction lots)
- 5/10/40 rule types

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Matching Models and Algorithms

Feature	Benefits	Enabler
• Multiple risk constraints • Multiple index tracking • Market impact/transaction costs • Soft constraint quadratic penalties • Variance or STD measure of risk • Uncertainty in parameter inputs	• Comprehensive risk control • Include realistic trading costs • Add flexibility to constraint management • Obtain stable portfolios • Reduce trading costs	**Second-Order Cone Algorithms**
• Min. holding/trading per stock • Max. number of holdings • Long/short-term capital gains/losses • If-then-else rules	• Effectively control size and number of positions • Better returns through after-tax implementation • Include dynamic strategy management	**Discrete Algorithms**

- Modern strategies require combination of both approaches
- Only exact methods provide solution quality assessment

MVO Variant: Maximize Rations (Sharpe Ratio, Information Ratio)

- Sharpe Ratio (SR), Information Ratio (IR) are popular performance measures:

$$SR(w) = \frac{\alpha^{\mathrm{T}} w}{\sqrt{w^T Q w}} \qquad IR(w,b) = \frac{\alpha^{\mathrm{T}}(w-b)}{\sqrt{(w-b)^T Q (w-b)}}$$

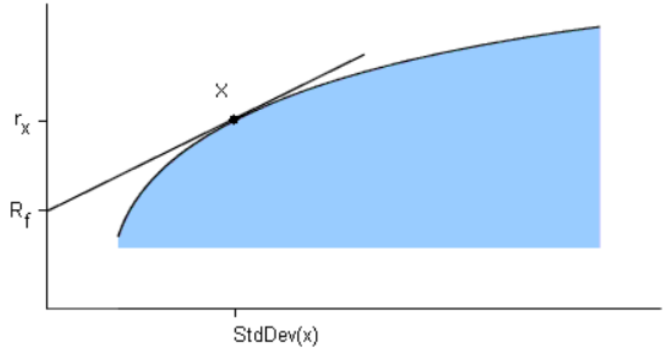
but unfortunately *non-convex* functions → so not suited for QP, SOCP,...

- Fortunately, they have one helpful property: scale invariance.

Portfolio scaled by any $\tau > 0$ leaves SR (and IR) unchanged:

$$SR(\tau \cdot w) = \frac{\alpha^{\mathrm{T}}(\tau \cdot w)}{\sqrt{(\tau \cdot w)^T Q (\tau \cdot w)}} = \frac{\tau \cdot \alpha^{\mathrm{T}} w}{\sqrt{\tau^2}\sqrt{w^T Q w}} = \frac{\alpha^{\mathrm{T}} w}{\sqrt{w^T Q w}} = SR(w)$$

Maximum Sharpe Ratio in the Efficient Frontier:



- Scale invariance allows to fix numerator: $\max \quad SR(w) = \frac{\alpha^{\mathrm{T}} w}{\sqrt{w^T Q w}} \quad \Leftrightarrow$

$$\begin{array}{ll} \min & \sqrt{(\tau \cdot w)^T Q (\tau \cdot w)} \\ s.t. & \alpha^{\mathrm{T}}(\tau \cdot w) = \pm 1 \\ & \tau > 0 \end{array}$$

 |

Maximizing Rations (SR, IR) – Reformulation Example

- Example 130-30 strategy: add 'scale asset' τ, substitute w by $\tilde{w} = \tau \cdot w$:

$$
\begin{array}{llrcl}
 & \max & SR(w) & = & \dfrac{\alpha^T w}{\sqrt{w^T Q w}} \\
\text{(long side)} & s.t. & \sum_{i \in U} w_i^+ & = & 130\% \\
\text{(net fully invested)} & & \sum_{i \in U} w_i & = & 100\% \\
\text{(sector bounds)} & & \sum_{i \in S_j} w_i & \le & 10\% \\
\text{(asset bounds)} & & w_i & \le & 3\% \\
\text{(\# shorts)} & & |\{i: w_i < 0\}| & \le & 20
\end{array}
$$

$\Leftrightarrow$

$$
\begin{array}{llrcll}
\min & \sqrt{\tilde{w}^T Q \tilde{w}} & & & & \\
s.t. & \sum_{i \in U} \tilde{w}_i^+ - \tau \cdot 130\% & & = & 0 & \\
 & \sum_{i \in U} \tilde{w}_i - \tau \cdot 100\% & & = & 0 & \\
 & \sum_{i \in S_j} \tilde{w}_i - \tau \cdot 10\% & & \le & 0 & \\
 & \tilde{w}_i - \tau \cdot 3\% & & \le & 0 & \\
 & |\{i: \tilde{w}_i < 0\}| & & \le & 20 & \\
 & & \alpha^T \tilde{w} & = & 1 & \text{(fix alpha)} \\
 & & \tau & \ge & 0.001 & \text{(scale)}
\end{array}
$$

(homogeneous reformulation)

- Solution: regain w by $w = \dfrac{\tilde{w}}{\tau}$

- Doesn't work for a few components (e.g., thresholds, tax-aware optimization)

Modeling Tricks – Alternative Turnover

- Portfolio Construction uses a variety of trading limitations (on net trades/buys/sells in any weighted aggregation, on transaction costs, on short sells, trade thresholds, and on two-sided turnover – our standard definition)

- Our libraries cover most one-/two-sided turnover conditions, either directly or as suitable reformulations.

  Example: total buys in a sector S ≤ 10% of all buys - modeled as buy constraint with special coefficients:

$$\sum_{i \in S} t_i^+ \leq 0.1 \cdot \sum_{i \in U} t_i^+ \quad \Longleftrightarrow \quad \sum_{i \in S} 0.9\, t_i^+ + \sum_{i \in U \setminus S} (-0.1)\, t_i^+ \leq 0$$

- But there are (and always will be) other turnover definitions as well, and users who want to model them, too….

  Example: in long-short portfolios with non-balanced trade lists, can you model min(buys, sells) ≤ M ?

  Unfortunately, we do not have a min function for constraints….so we need a modeling trick…

  It holds that $min(a, b) = \dfrac{a+b}{2} - \dfrac{|a-b|}{2}$ and we have linear and absolute value functions!

  We can also define auxiliary variables and use one-sided constraints to capture the sum of all buys and of all sells. With these in combination, we can then derive an exact modeling for limiting the minimum of all buys and sells.

Modeling Tricks – Limit Top n Positions (Intro)

- Portfolio Construction uses a variety of holding limitations (on net/long/short/absolute holdings in any aggregation, on active/total holdings in any aggregation, on long-short ratios, holdings and exposure levels, etc.)

- It also uses conditional holding limitations, e.g. 5/10/40 rule which is critical for UCITS compliance:

$$w^{I_k} := \sum_{i \in I_k} w_i \leq 10\% \quad \forall k = 1,\dots,K$$

(not more than 10% holding in any issuer)
= limit on single issuer concentration

$$\sum_{k: w^{I_k} \geq 5\%} w^{I_k} \leq 40\%$$

(all issuers on or above 5% make at most 40% of portfolio)
= limit on portfolio portion in concentrated form

- Due to their practical relevance, we have implemented such conditional constraints.

  This flexibility clearly raises appetite for more…

- One popular request is: Can you limit the top n position weight sum in the portfolio? (No threshold specified!)

- Our popular answer is: Yes, you can. Though this is really getting you deep into optimization theory…

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Modeling Tricks – Limit Top n Positions (Modeling Part I)

- The modeling is not easy to explain....but luckily will be easy to set up.

  For those interested to take a closer look:

  G. Cornuejols, R. Tütüncü: *Optimization Methods in Finance*
  D. Bertsimas, J.Tsitsiklis: *Introduction to Linear Optimization*
  M. Padberg: *Linear Optimization and Extensions*

- We restrict here to a long-only case: $w_i \geq 0$ (can be extended to long-short).
- Let's start with an assumption (which will become superfluous at the end):

  We magically know the optimal portfolio weights w_i
  and consider them as known parameters rather than variables.

- Then a way to determine the top N positions is to solve this LP for the binary decision variables z_i

$$\max \sum_{i=1,\dots,N} w_i \cdot z_i \qquad = \text{top n positions weight}$$

(LP1)

$$
\begin{aligned}
\text{s.t.} \quad & \sum_{i=1,\dots,N} z_i = n & (q)\\
& z_i \leq 1 \quad \forall i = 1,\dots,N & (p_i)\\
& 0 \leq z_i \quad \forall i = 1,\dots,N &
\end{aligned}
$$

dual variables

Key observation:

(LP1) is "totally unimodular"

= solution will be integral (i.e., binary), without explicit condition!

Modeling Tricks – Limit Top n Positions (Modeling Part II)

- Each LP has a dual LP, in this case (DP1):

(LP1)
$$\begin{aligned}
\max \quad & \sum_{i=1,\dots,N} w_i \cdot z_i \\
\text{s.t.} \quad & \sum_{i=1,\dots,N} z_i = n \\
& z_i \le 1 \qquad \forall i = 1,\dots,N \\
& 0 \le z_i \qquad \forall i = 1,\dots,N
\end{aligned}$$

(DP1)
$$\begin{aligned}
\min \quad & n \cdot q + \sum_{i=1,\dots,N} p_i \\
\text{s.t.} \quad & q + p_i \ge w_i \qquad \forall i = 1,\dots,N \\
& p_i \ge 0 \qquad \forall i = 1,\dots,N
\end{aligned}$$

- From duality theory, we know for the objective functions:

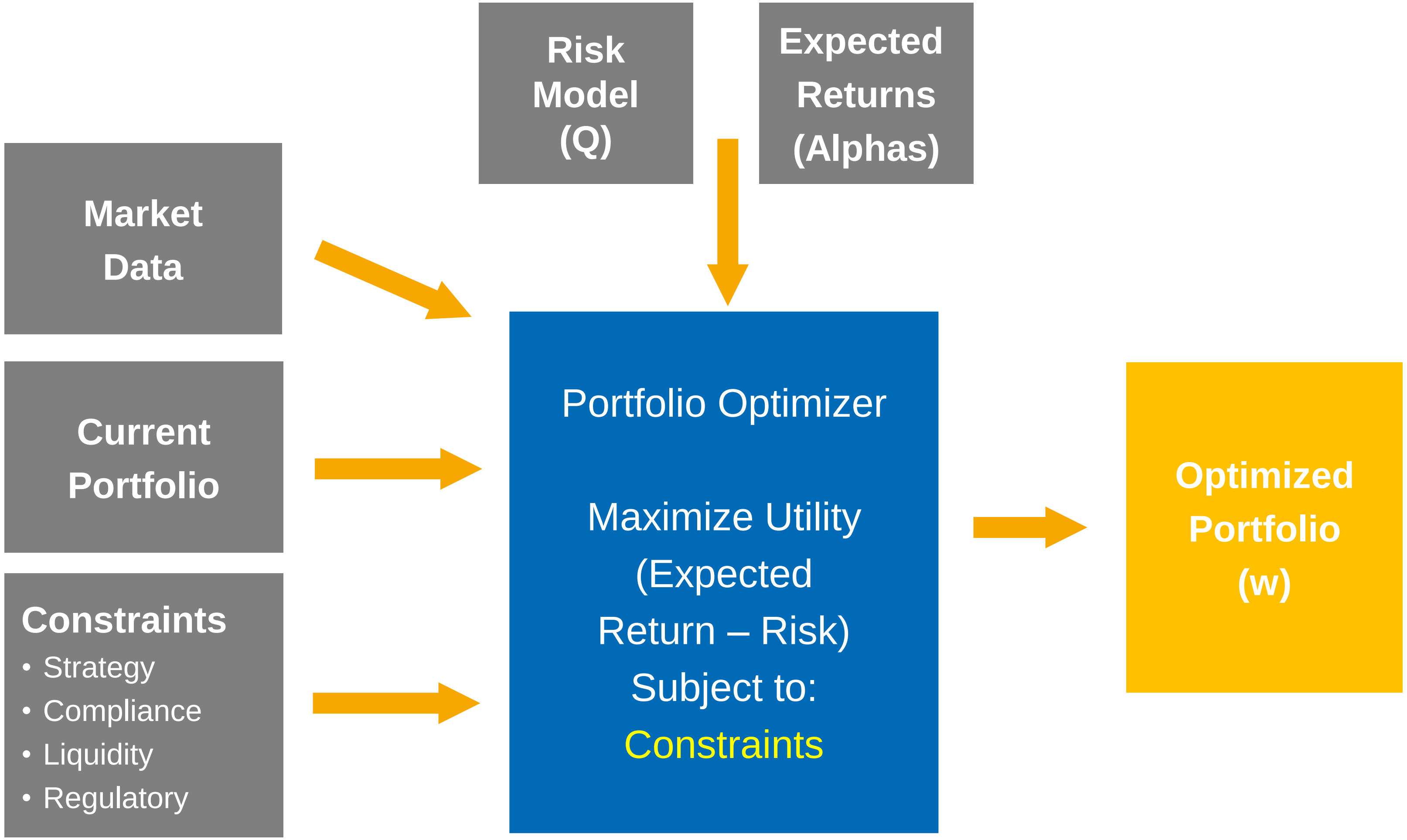
Top n positions weight = (LP1) = $\sum_{i=1,\dots,N} w_i \cdot z_i \le n \cdot q + \sum_{i=1,\dots,N} p_i$ = (DP1)

- So it suffices to limit the dual objective function (and add the dual constraints)!
- Result: A top n position limit L can be modeled with auxiliary variables p_i, q:

$$\begin{aligned}
n \cdot q + \sum_{i=1,\dots,N} p_i &\le L \\
q + p_i \ge w_i & \qquad \forall i = 1,\dots,N \\
p_i \ge 0 & \qquad \forall i = 1,\dots,N
\end{aligned}$$

$\leftarrow$ Here, we don't need to assume anymore that the weights w_i are known!
So they become variable(s) again!

Constraint Attribution:
How do constraints affect the optimized portfolio?



Which effect do constraints have on the solution (esp. re alpha)?

Implied alpha
= alpha to use in an unconstrained problem to obtain same optimal portfolio:

$$\alpha^* = 2\lambda Q w^*$$

So:
difference alpha / implied alpha
= constraint effect!

Constraint Attribution – Optimization Principles

$$\overbrace{\begin{array}{lll} \text{Maximize} & \alpha^{\mathrm{T}} w - \lambda w^{\mathrm{T}} Q w & \\ \text{Subject to} & e^{\mathrm{T}} w = 1 & (\pi) \\ & w \geq 0 & (\mu) \end{array}}^{\text{Optimization Problem}} \quad \overbrace{\begin{array}{l} \alpha - 2\lambda Q w - \pi e + \mu = 0 \\ \mu_i w_i = 0 \\ \mu \geq 0 \end{array}}^{\text{Optimality Conditions}}$$

- Optimality conditions can be reformulated and beneficially interpreted:

$$\alpha^* = 2\lambda Q w^* = \alpha - \pi e + \mu$$

Implied Alpha Decomposition

$$w^* = \frac{1}{2\lambda} Q^{-1} \alpha - \frac{1}{2\lambda} \pi Q^{-1} e + \frac{1}{2\lambda} Q^{-1} \mu$$

Holdings Decomposition

$$\alpha^{\mathrm{T}} w^* = \frac{1}{2\lambda} \alpha^{\mathrm{T}} Q^{-1} \alpha - \frac{1}{2\lambda} \alpha^{\mathrm{T}} \pi Q^{-1} e + \frac{1}{2\lambda} \alpha^{\mathrm{T}} Q^{-1} \mu$$

Expected Return Decomposition

- Constraint Attribution provides solution sensitivities vs. constraints!
- Of course tricky to extend into combinatorial/non-continuous-convex context!

Robust Optimization in Finance

“ Planning is the replacement of chance by error ”

…so let’s see how we can control the error!

Science vs. Intuition: Do You Select The Right Assets?

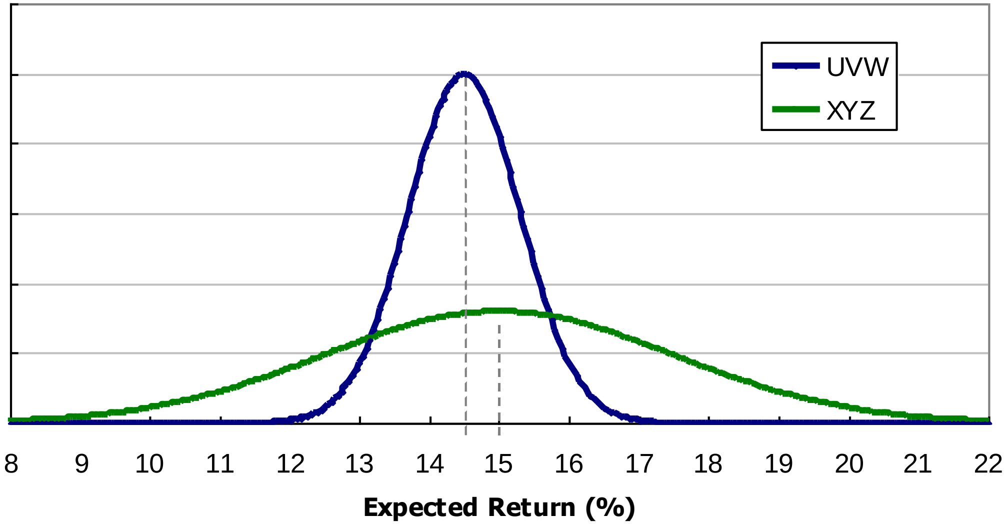
- Assume you are indifferent (from a risk perspective) between the two assets XYZ and UVW.

  They have the following return expectations:

	Expected Return
XYZ	15.0%
UVW	14.5%

  How would you weight these two assets in the optimal portfolio?

- Would you make the same choice if you knew these return *distributions*?



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Changing Inputs and Portfolios

- Let's do another experiment: comparing outcomes of "almost same problems":

	α^1	α^2	σ
Asset 1	7.15%	7.16%	20%
Asset 2	7.16%	7.15%	24%
Asset 3	7.00%	7.00%	28%

alphas shifted by only 1bp

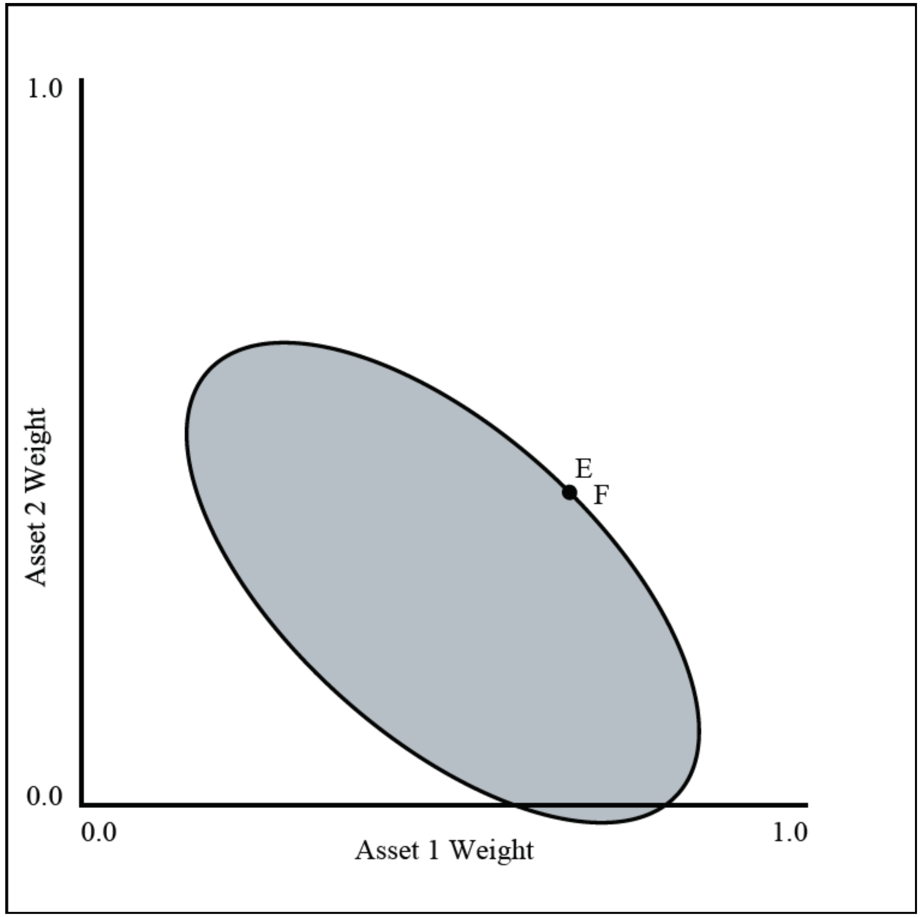
- We use these inputs in different investment approaches:

  - Long-short investing — Both get a fixed risk limit.
  - Long-only investing — Long-only also gets non-negativity and a budget constraint.

Changing Inputs and Portfolios

- **Long-Short Solution:**
  - Grey zone
    = feasibility area (Asset 1, Asset 2 space)
  - Elliptic due to fix risk budget
  - Shorting permitted
    (weights can be negative)
  - Optimal portfolios (E, F) are found
    in the boundary
  - Small shift in alphas (1bps) results in
    minor change of portfolio weights (<10bps).

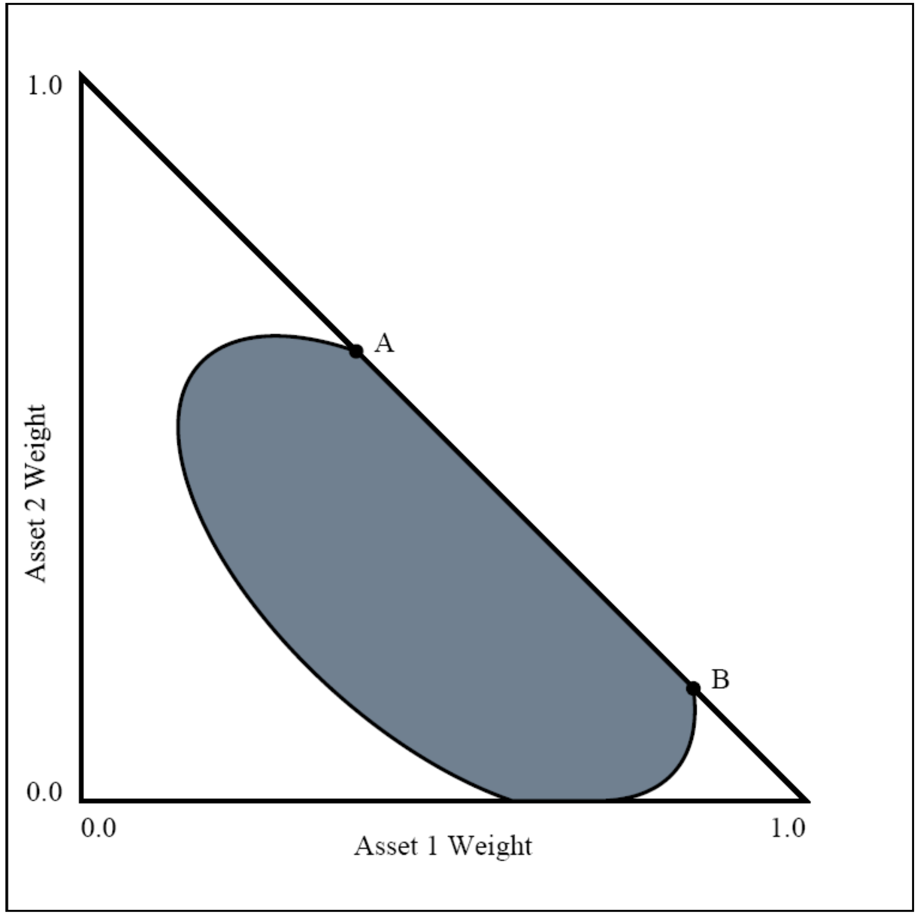
	Portfolio E	Portfolio F
Asset 1	67.18%	67.26%
Asset 2	43.10%	43.01%
Asset 3	-10.28%	-10.28%



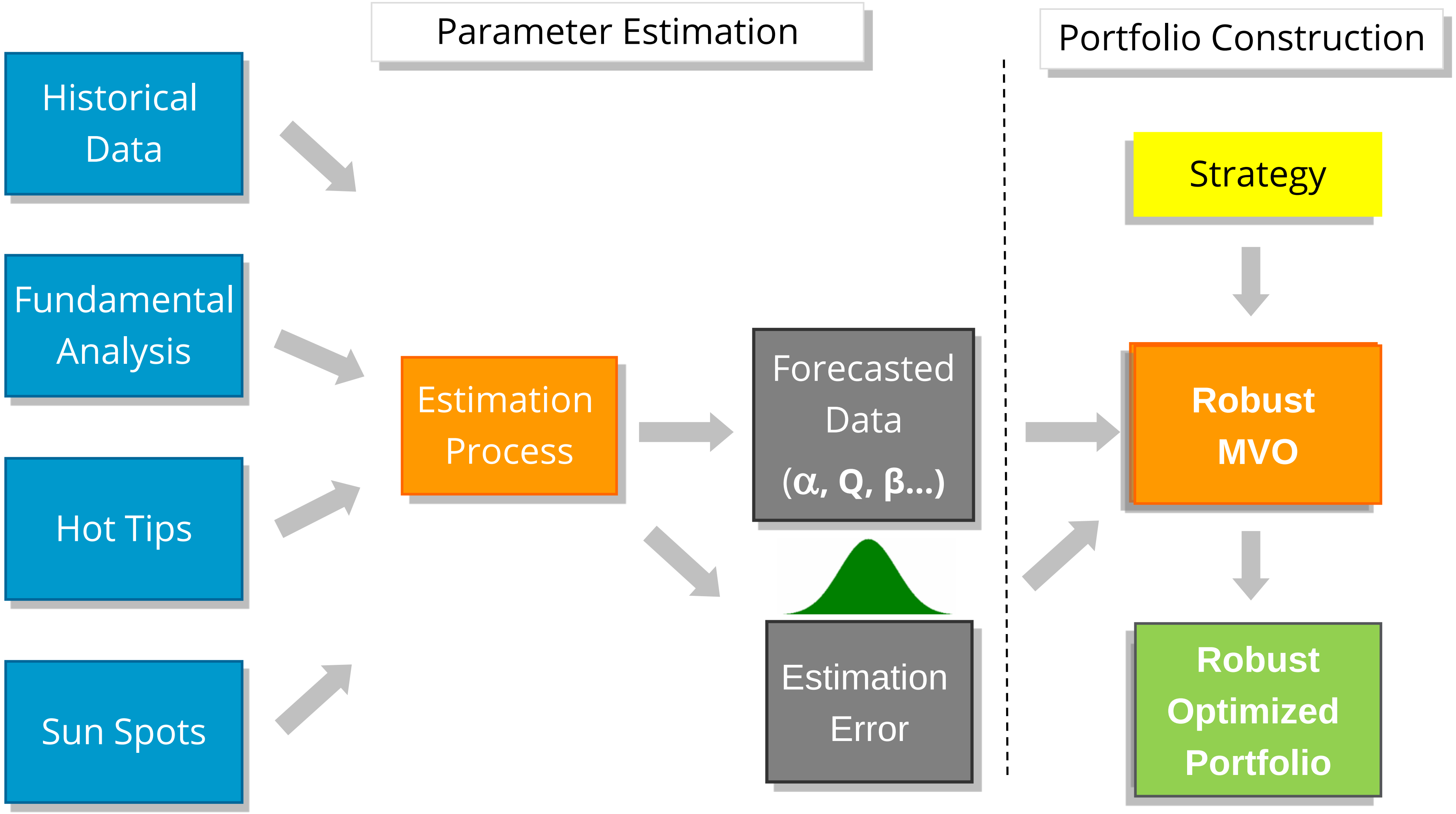
Changing Inputs and Portfolios

- **Long-Only Solution:**
  - Grey zone
    = feasibility area (Asset 1, Asset 2 space)
  - Elliptic due to fix risk budget. No shorting.
  - 45° cut due to budget constraint.
  - Optimizer always looks for corners.
    Constraints create corners…
  - Optimal portfolios (A, B) are found
    in corners now!
  - Small shift in alphas (1bps) results in
    huge change of portfolio weights (~50%).

	Portfolio A	Portfolio B
Asset 1	38.1%	84.3%
Asset 2	61.9%	15.7%
Asset 3	0.0%	0.0%



The Proposed Solution: Integrating Estimation Process and Robust MVO



"When solving for the efficient portfolios, the **differences in precision of the estimates** should be **explicitly incorporated** into the analysis"

H. Markowitz

Robust MVO explicitly uses information on distributions of forecasted returns (estimation error) to find **robust portfolios**

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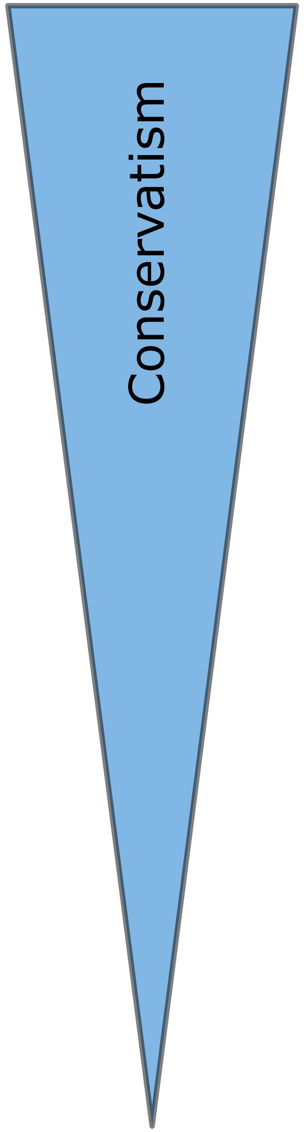
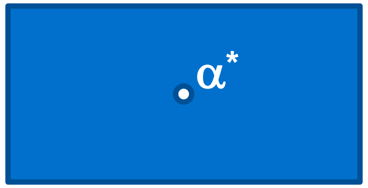
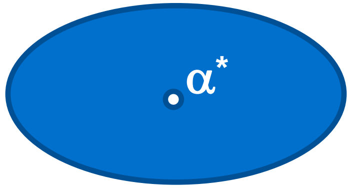
Uncertainty Regions



- Box uncertainty region:
  - independent alpha estimation errors
  - linear model



- Elliptic uncertainty region:
  - alpha estimation errors are coupled
  - SOCP model

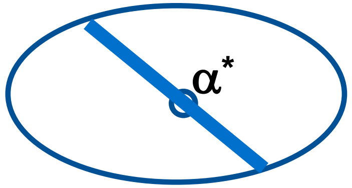


- Zero-net alpha adjustment approach:
  - additional condition:

$$e^T D(\alpha^* - \alpha) = 0$$

  - alpha adjustments sum up to 0
  - modeled with specific prior portfolio:

$$k \left\| \Sigma^{1/2} \left(w - \frac{e^T D \Sigma w}{e^T D \Sigma D^T e} D^T e \right) \right\|$$



(D=I)

Maximizing Robust Returns

How do we maximize Robust Expected Returns?

- Assume the vector of expected returns is $\alpha \sim N(\alpha^*, \Sigma)$
- Define an elliptical confidence region around the mean estimated expected returns α^* as

$$\{\alpha : (\alpha - \alpha^*)^T \Sigma^{-1} (\alpha - \alpha^*) \leq k^2\}$$

(if the errors are normally distributed, k^2 comes from the χ^2-distribution)

- **Robust Objective**: The optimization problem is defined as:

$$\textbf{Max (Min E(return))} = \textbf{Max}_w \, \textbf{Min}_{\alpha \in B(\alpha^*)} \, \mathbf{E}(\alpha w)$$

- $B(\alpha^*)$ is region around α^* taken into account as potential errors in the estimates of expected returns

"A **robust objective adjusts estimated expected returns** to counter the (negative) effect that optimization has **on the estimation errors** that are present in the estimated expected returns"

Robust MVO Can Reduce Overestimation/Underestimation

Robust MVO frontiers are closer to the true frontier

Estimated Robust Frontier

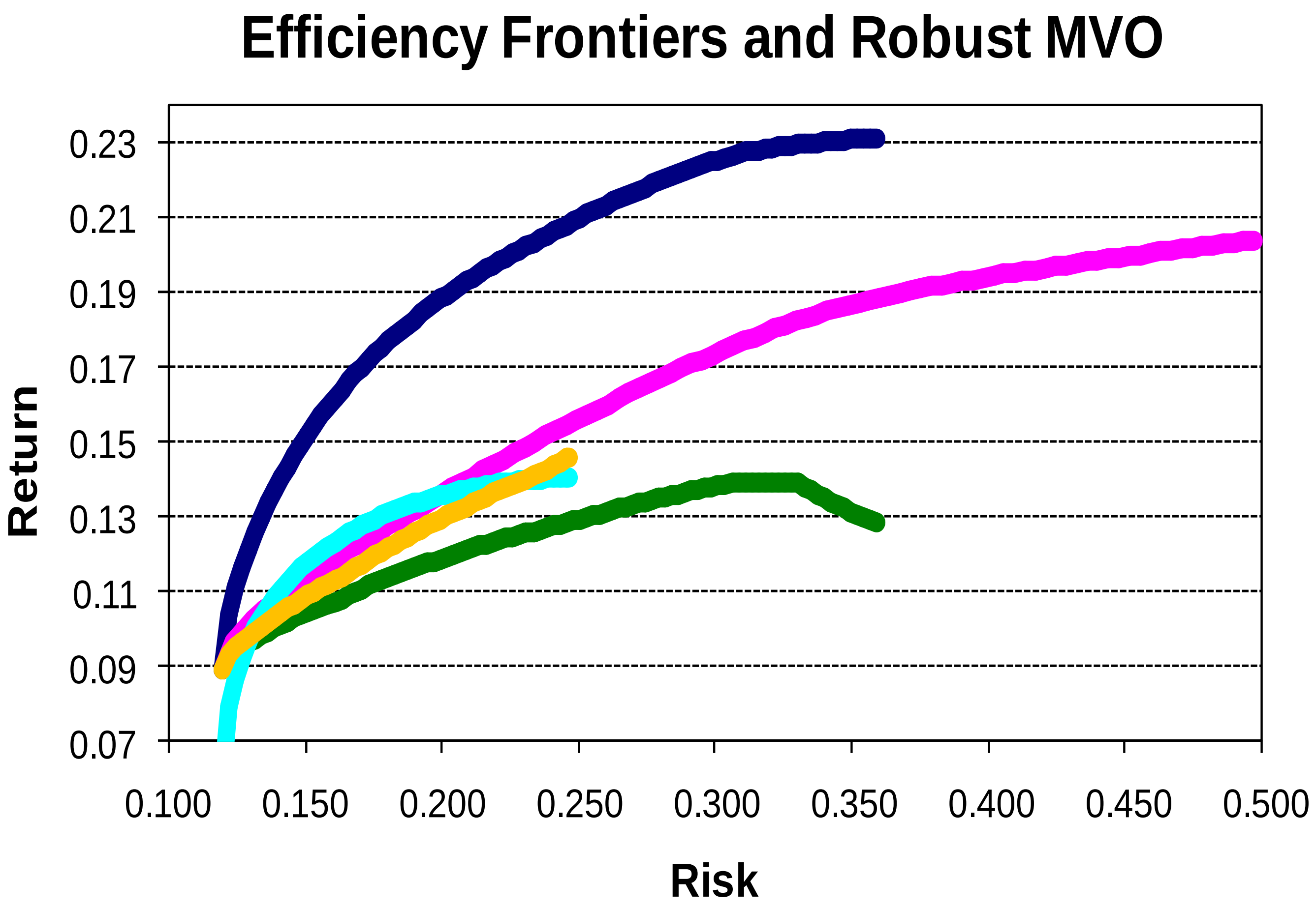
Robust Efficiency Frontier computed using the estimated expected returns and the estimation error

True Frontier

Efficiency Frontier computed using the true future returns

Actual Robust Frontier

Return for the portfolios in the Estimated Robust Frontier using the true expected returns



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Multi-Period Optimization: Optimize Portfolio Evolutions

- Portfolio model with several stages (decision points) loosely coupled together.

  First stage represents today's optimization, and
  stages two and beyond represent future optimizations.

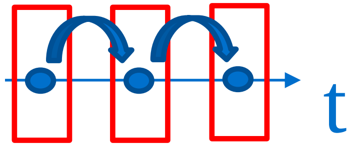
  Length of first stage represents the rebalancing time-horizon.

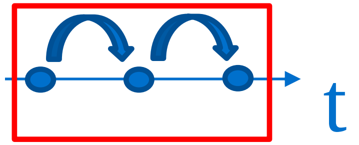
- Make today's decisions aware of expected future circumstances!

- We only implement today's decision. After the current period expires, we "resolve" the optimization (rolling horizon).

- Multi-Period Optimization models have applications in:
  - Quantitative investing with various multi-horizon alphas
  - Trade scheduling
  - Dynamic benchmark tracking around reconstitutions
  - Taxable optimization
  - ….

Backtest
(simulation through time)





Evolution Optimization
(optimize all periods at once)

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Multi-Period Optimization: Portfolio Evolutions Model

- $$\max_{x^t,\Delta x^t} \quad \sum_{t=1}^{T} (\alpha^t)^T x^t - \delta \mathrm{TC}(\Delta x^t) - \gamma (x^t)^T Q x^t$$

  subject to

  $$\begin{aligned} x^{t+1} &= (1+r^t)x^t + \Delta x^{t+1} + p^t, && t = 0, \dots, T-1 \\ z^{t+1} &= r^t x^t + z^t + c^t, && t = 0, \dots, T-1 \\ (x^t, \Delta x^t) &\in \Omega_t, && t = 1, \dots, T \end{aligned}$$

- The T stages represent points in time at which decisions are taken (e.g., portfolio rebalancings, trades, …).
- x^t, Δx^t, and z^t denote the holdings, trades, and portfolio size in period t.
- p^t and c^t represent external position and cash injections/removals.
- Objective trades off return, TC, and risk across periods.
- First two equation sets "couple" the stages together (holdings and portfolio size, including pot. cash flows).
- Final set of constraints (Ω_t) represent the "stage" constraints – conditions to hold for individual portfolios.
- Inputs include alpha, TC and risk model, "roll-forward" estimates, initial portfolio, and cash inflows for every stage.

Deterministic model – would be nice to get it combined with more stochastics….

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In this talk, we address:

1. **How do practitioners use optimization in finance?**
   - Introduction & Terms (Darling, look – this is my alpha!)
   - Modeling risk
   - Investment strategies

2. **Which practical needs are tackled with which techniques?**
   - The clue of Markowitz - Mean-Variance Optimization (and variants)
   - Exploiting estimation errors - Robust Optimization
   - Conquering dimension time - Multi-Period Optimization

3. **Which problems are still out there for optimization research?**
   - Round-lotting etc - General mixed-integer problems
   - Estimation errors in risk models – Semi-definite programming
   - Trade Scheduling, High Frequency Trading – Online optimization

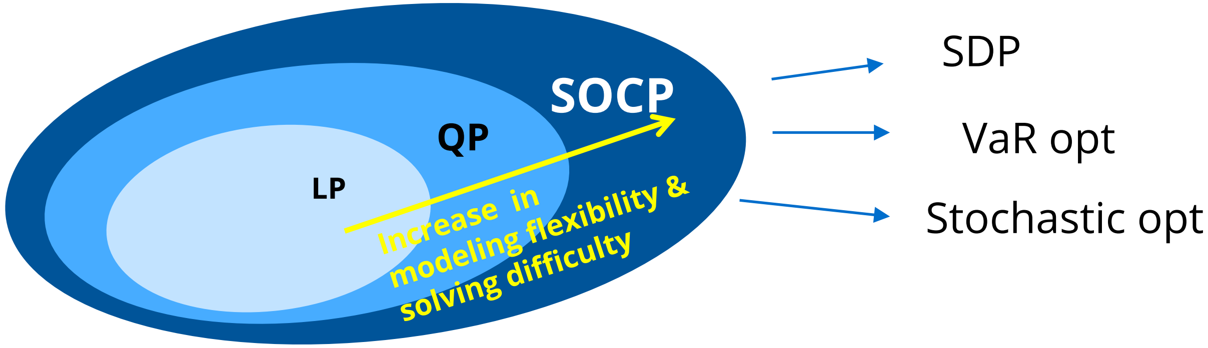
n. **So…who did really shrink the alpha?**
   - Economy
   - Politics
   - Corona
   - George Dantzig
   - Ben-Tal & Nemirovski
   - Black & Litterman
   - Reality
   - The optimizer
   - Darling

Solvability Challenges

- Combinatorial complexity:
  - only combinatorial structures invoke discrete overlay method (B&B) (smooth continuous SOCP problems are solved optimal in one iteration!)
  - any such structures hamper solvability, but they are of different harm
  - combinatorial is everything involving discrete decisions (yes/no, integral, conditional)

    in detail:
    - thresholds
    - counts (names, trades)
    - conditional constraints
    - one-sided conditions and absolute values (min)
    - pw lin tcost
    - fixed charge
    - tax
  - very tricky: round-lotting – trades only in discrete amounts (e.g., x * 100 shares)
- Symmetry:
  - multiple solutions with similar properties exist – hard to improve dual bound
  - poisonous for optimizer (mostly for optimality, less so feasibility)
- Tightness:
  - tight constraints can be helpful (guiding towards close to optimal solutions)
  - but can also be problematic – if it is hard to find feasible solutions at all!
- Degeneracy:
  - means that more than (dimensionally) necessary constraints are binding
  - usually hard to identify and typically not a big issue

Generalized Methodologies



- Semi-Definite Programming (SDP):
  - Robust MVO can handle (elliptic) uncertainty around linear terms (α, β, …)
  - Risk models are estimators as well – and come with σ uncertainty
  - Using "uncertain" risk models requires SDP techniques solvable for 10,000's of assets
- Optimizing higher moments of distributions:
  - MVO accounts only for first (mean) and second (std.dev.) moment of return distributions
  - Skewness and kurtosis control for non-Gaussian distributions desired – difficult to implement
- Stochastic optimization:
  - VaR optimization (CVaR can be approached linearly, WCVaR is promising approach)
  - Non-linear payout structures (e.g., options)
  - Multi-period: Aging of portfolios (different progression paths, e.g., expiring options, used optionalities in instruments, like convertible bonds)

Online Optimization for Trading

- Finance world never stops – natural Online Optimization problems
- Trade execution problems:
  - Trade scheduling: Execute large trades in multiple tranches – when best to trade how much? (Supply/demand, market impact, tcosts, risk…)
  - Building or unwinding entire portfolios
  - Transition management – transform one portfolio into a defined target portfolio, in multiple steps
- High frequency trading:
  - Algorithms trade automatically
  - Usually executed on very liquid assets (e.g. bluechip stocks) or instruments with very volatile pricing (e.g., futures priced in ticks)
  - Trying to leverage trade needs of others: find large trade placements and be "on other side" (exploit market impact others suffer as gain)
  - In effect: trying to exploit trading signals (= very short-term alpha) in trading patterns

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In this talk, we address:

1. **How do practitioners use optimization in finance?**
   - Introduction & Terms (Darling, look – this is my alpha!)
   - Modeling risk
   - Investment strategies
2. **Which practical needs are tackled with which techniques?**
   - The clue of Markowitz - Mean-Variance Optimization (and variants)
   - Exploiting estimation errors - Robust Optimization
   - Conquering dimension time - Multi-Period Optimization
3. **Which problems are still out there for optimization research?**
   - Round-lotting etc - General mixed-integer problems
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n. **So…who did really shrink the alpha?**

- Economy
- Politics
- Corona
- George Dantzig
- Ben-Tal & Nemirovski
- Black & Litterman
- Reality
- The optimizer
- Darling

So….who finally shrunk the alpha?

- Realized alpha (= materialized return) is usually shrunk by:
  - Wrong predictions, bad bets, bad portfolio managers, …
  - Risk aversion of investors – not trusting their own views
  - Darling (reducing potential alpha by using capital for shopping instead)
  - Adverse economic environments, e.g., “the unexpected bear”

The unexpected bear

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  - Adverse economic environments, e.g., “the unexpected bear”
  - Geopolitical trouble

Watch it, shorty!

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  - Adverse economic environments, e.g., "the unexpected bear"
  - Geopolitical trouble especially when mixed with economic issues
- But alpha can even be expectedly shrunk in the planning phase :
  - Robust MVO can be considered a shrinkage approach

Robust MVO: Shrinkage Interpretation

Adjustment of the return expectation α^* is

$$\alpha = \alpha^* - \underbrace{\sqrt{\frac{k^2}{w^T \Sigma w}}}_{\text{shrinkage factor towards market prior}} \Sigma w$$

where w is the market portfolio,

Σ is the asset covariance matrix

k is the “estimation error aversion” multiplier

For simplicity, assume Σ is a diagonal covariance matrix and we have a long-only portfolio:

- For positive weighted assets , the adjustment factor will be positive which will decrease the alpha from α^*
- For negative weighted assets, the adjustment factor will be negative which will increase the alpha from α^*

“One way to describe Robust Portfolio Optimization is that it seeks to **identify the maximum possible forecast error** and then select the portfolio that best **minimizes the risk of making that error**. In other words, the emphasis is not on choosing the portfolio with the highest expected return, but rather the one likely to produce the **highest possible return even if the return forecasts are wrong**."

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- But alpha can even be expectedly shrunk in the planning phase:
  - Robust MVO can be considered a shrinkage approach
  - Black-Litterman: use non-quantified return predictions efficiently – shrinking alpha views towards a market prior
  - Constraints prevent full translation of predicted alpha into portfolios – “operational shrinkage” through optimizer

- **Conclusion**: who really shrinks the alpha:
  - Economy
  - Politics
  - Corona
  - George Dantzig
  - Ben-Tal & Nemirovski
  - Black & Litterman
  - Reality
  - The optimizer
  - Darling (?)

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Many thanks for your attention!

Further questions?

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