A Gentle and Incomplete Introduction to Bilevel Optimization
… and Some New Results

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Agenda

1. What is bilevel optimization anyway?
2. Some theory of linear bilevel problems
3. How do you solve a linear bilevel problem?
4. About yesterday
5. The burial of coupling constraints in linear bilevel optimization
6. Connections between robust and bilevel optimization
7. An (in my opinion) important open problem

There should be no crying in this lecture!

I will (try to) teach principles, not formulas!

You will not remember the last ε,
but I hope you remember the core ideas!

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but I hope you remember the core ideas!

Interrupt me and ask questions at any time!

Background Material: Our Upcoming Book

Linear and Mixed-Integer Bilevel Optimization: Theory and Algorithms

Jointly with Ivana Ljubic and Yasmine Beck

Free pre-publication version

What is bilevel optimization anyway?

“Usual” single-level problems

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & f(x) \\ \text{s.t.} \quad & g(x) \geq 0 \\ & h(x) = 0 \end{aligned}$$

- only one objective function f
- one vector of variables x
- one set of constraints g and h

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This models a situation in which a single decision maker takes all decisions, i.e., decides on the variables of the problem.

Very often, that’s appropriate:

- a single dispatcher controls a gas transport network
- a single investment bank decides on the assets in a portfolio
- a single logistics company decides on its supply chain

Often, life's different

- Many situations in our day-to-day life are different
- Often:
    - A decision maker makes a decision ...
    - ... while anticipating the (rational, i.e., optimal) reaction of another decision maker
    - The decision of the other decision maker depends on the first decision
- Thus: the outcome (or in more mathematical terms, the objective function and/or feasible set) depends on the decision/reaction of the other decision maker

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Formalizing this situation leads to hierarchical or bilevel optimization problems

Informal example: Pricing

- A very rich class of applications of bilevel optimization
- First decision maker (leader)
    - decides on a price of a certain good (or maybe on different prices for multiple goods)
    - goal: maximize revenue from selling these goods

Informal example: Pricing

- A very rich class of applications of bilevel optimization
- First decision maker (leader)
    - decides on a price of a certain good
      (or maybe on different prices for multiple goods)
    - goal: maximize revenue from selling these goods
- Second decision maker (follower)
    - decides on purchasing the goods of the leader to generate some utility

Thus, ...

- the leader's decision depends on the optimal reaction of the follower
- the decision of the follower depends on the (pricing) decisions of the leader

Anti Drug Smuggling

- Graph models the network of drug smuggling routes
- Smugglers want to maximize the flow of drugs from an origin to a destination

Anti Drug Smuggling

- Graph models the network of drug smuggling routes
- Smugglers want to maximize the flow of drugs from an origin to a destination

- Follower: maximum flow (= amount of drugs) problem
- Leader: Interdiction of certain parts of the drug smuggling routes
- Goal of the leader: minimize the maximum flow
- Leader only has a certain budget
- … and maybe incomplete information about the follower’s problem

Anti Drug Smuggling

Canada and the Transcontinental Drug Links

Strategic Forecasting Inc
go to original

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Canadian police conducted several simultaneous raids on suspected drug traffickers in Newfoundland and Quebec provinces Oct. 11, arresting two dozen people and seizing marijuana, cocaine, weapons, cash and property. The drug-trafficking ring, which Canadian authorities believe was operated by the Quebec-based Hell's Angels motorcycle/crime gang, could have smuggled the cocaine into Canada from South America via Mexico and the United States.

The jungles of South America, where cocaine is produced, seem a long way from the St. Lawrence River. Using a sophisticated shipment and distribution network, however, criminal and militant organizations can cover the distance in a few days.

More than 70 members of the Royal Newfoundland Constabulary and Quebec's Provincial Biker Enforcement Unit carried out the raids, which represented the culmination of an 18-monthlong investigation dubbed Operation Roadrunner. The arrests were made near St. John's in Newfoundland and near the towns of Laval and La Tuque in Quebec. In Newfoundland, authorities seized $300,000 in cash, 51 pounds of marijuana and 19 pounds of cocaine, as well as vehicles, weapons and computers. In Quebec, $170,000 and four houses were seized.

A bit more formal, please

Definition (Bilevel optimization problem)

A bilevel optimization problem is given by

$$\begin{aligned} \min_{x \in X,\, y} \quad & F(x, y) \\ \text{s.t.} \quad & G(x, y) \geq 0 \\ & y \in S(x) \end{aligned}$$

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$S(x)$: set of optimal solutions to the x-parameterized problem

$$\begin{aligned} \min_{y \in Y} \quad & f(x, y) \\ \text{s.t.} \quad & g(x, y) \geq 0 \end{aligned}$$

A bit more formal, please ... continued

$$\begin{aligned} \min_{x \in X,\, y} \quad & F(x, y) \\ \text{s.t.} \quad & G(x, y) \geq 0 \\ & y \in S(x) \end{aligned}$$

... and ...

$$S(x) = \arg\min_{y \in Y} \{f(x, y) : g(x, y) \geq 0\}$$

Wording

- First problem: so-called upper-level (or the leader's) problem
- Second problem is the so-called lower-level (or the follower's) problem
- Both problems are parameterized by the decisions of the other player
- $x \in \mathbb{R}^{n_x}$: upper-level variables
    - decisions of the leader
- $y \in \mathbb{R}^{n_y}$: lower-level variables
    - decisions of the follower

A bit more formal, please ... continued

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Functions and dimensions

- Objective functions
  - $F, f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \to \mathbb{R}$
- Constraint functions
  - $G : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \to \mathbb{R}^m$
  - $g : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \to \mathbb{R}^\ell$
  - The sets $X \subseteq \mathbb{R}^{n_x}$ and $Y \subseteq \mathbb{R}^{n_y}$ are often used to denote integrality constraints.
  - Example: $Y = \mathbb{Z}^{n_y}$ makes the lower-level problem an integer program

A bit more formal, please ... continued

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Definition

1. We call upper-level constraints $G_i(x, y) \geq 0$, $i \in \{1, \ldots, m\}$, coupling constraints if they explicitly depend on the lower-level variable vector y.
2. All upper-level variables that appear in the lower-level constraints are called linking variables.

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Optimal-value function reformulation

Instead of using the point-to-set mapping S ... one can also use the so-called optimal-value function

$$\varphi(x) := \min_{y \in Y} \{f(x,y) : g(x,y) \geq 0\}$$

and re-write the bilevel problem as

$$\begin{aligned} \min_{x \in X, y \in Y} \quad & F(x,y) \\ \text{s.t.} \quad & G(x,y) \geq 0, \; g(x,y) \geq 0 \\ & f(x,y) \leq \varphi(x) \end{aligned}$$

Shared constraint set, bilevel feasible set, inducible region

Definition

The set

$$\Omega := \{(x, y) \in X \times Y \colon G(x, y) \geq 0,\ g(x, y) \geq 0\}$$

is called the shared constraint set.

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Definition

The set

$$\mathcal{F} := \{(x,y) \colon (x,y) \in \Omega,\ y \in S(x)\}$$

is called the bilevel feasible set or inducible region.

Single-level relaxation

Definition

The problem of minimizing the upper-level objective function over the shared constraint set, i.e.,

$$\begin{aligned} \min_{x,y} \quad & F(x,y) \\ \text{s.t.} \quad & (x,y) \in \Omega, \end{aligned}$$

is called the single-level relaxation (SLR) of the bilevel problem.

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Remark

- The single-level relaxation is identical to the original bilevel problem except for the constraint $y \in S(x)$, i.e., except for the lower-level optimality.
- Thus, it is indeed a relaxation.

Pricing revisited

- First bilevel pricing problem with linear constraints, linear upper-level objective, and bilinear lower-level objective: Bialas and Karwan (1984)

Pricing revisited

- First bilevel pricing problem with linear constraints, linear upper-level objective, and bilinear lower-level objective: Bialas and Karwan (1984)
- Here: a more general version taken from Labbé et al. (1998)

$$\begin{aligned}
\max_{x, y=(y_1,y_2)} \quad & x^\top y_1 \\
\text{s.t.} \quad & Ax \leq a \\
& y \in \arg\min_{\bar{y}} \left\{ (x + d_1)^\top \bar{y}_1 + d_2^\top \bar{y}_2 : D_1 \bar{y}_1 + D_2 \bar{y}_2 \geq b \right\}
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Pricing revisited

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- Vector y of lower-level variables is partitioned into two sub-vectors y_1 and y_2, called plans, that specify the levels of some activities such as purchasing goods or services

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- Upper-level player influences the activities of plan y_1 through the price vector x that is additionally imposed onto y_1

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- Price vector x is subject to linear constraints that may, among others, impose lower and upper bounds on the prices

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    - These activities may, e.g., be substitutes offered by competitors for which prices are known and fixed

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- The lower-level player determines his activity plans y_1 and y_2 to minimize the sum of total disutility and the price paid for plan y_1 subject to linear constraints
- To avoid the situation in which the leader would maximize her profit by setting prices to infinity for these activities y_1 that are essential, one may assume that the set $\{y_2 : D_2 y_2 \geq b\}$ is non-empty

Anti Drug Smuggling Revisited

Follower: w-parameterized maximum flow problem

$$
\begin{aligned}
\varphi(w) := \max_{f \in \mathbb{R}^{|A|}} \quad & \sum_{a \in \delta^{\text{out}}(s)} f_a - \sum_{a \in \delta^{\text{in}}(s)} f_a \\
\text{s.t.} \quad & \sum_{a \in \delta^{\text{out}}(v)} f_a - \sum_{a \in \delta^{\text{in}}(v)} f_a = 0, \quad v \in V \setminus \{s, t\} \\
& f_a \geq 0, \quad a \in A \\
& f_a \leq c_a(1 - w_a), \quad a \in A
\end{aligned}
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& f_a \leq c_a(1 - w_a), \quad a \in A
\end{aligned}
$$

Leader: Maximum flow interdiction

$$
\begin{aligned}
\min_{w \in \{0,1\}^{|A|}} \quad & \varphi(w) \\
\text{s.t.} \quad & \sum_{a \in A} w_a \leq B
\end{aligned}
$$

An Academic and Linear Example (Kleinert 2021)

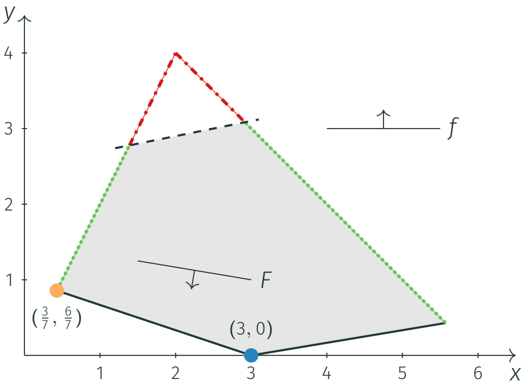
Upper-level problem

$$\begin{aligned}
\min_{x,y} \quad & F(x,y) = x + 6y \\
\text{s.t.} \quad & -x + 5y \leq 12.5 \\
& x \geq 0 \\
& y \in S(x)
\end{aligned}$$

Lower-level problem

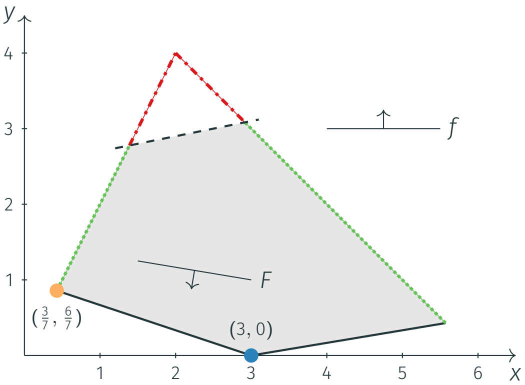
$$\begin{aligned}
\min_{y} \quad & f(x,y) = -y \\
\text{s.t.} \quad & 2x - y \geq 0 \\
& -x - y \geq -6 \\
& -x + 6y \geq -3 \\
& x + 3y \geq 3
\end{aligned}$$

An Academic and Linear Example (Kleinert 2021)



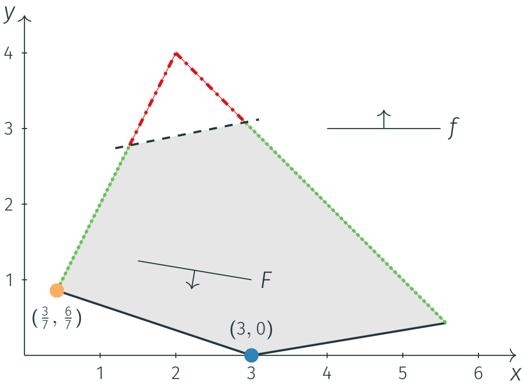
- Shared constrained set: gray area
- Green and red lines: nonconvex set of optimal follower solutions (lifted to the x-y-space)
- Green lines: Nonconvex and disconnected bilevel feasible set of the bilevel problem

An Academic and Linear Example (Kleinert 2021)



1. The feasible region of the follower problem corresponds to the gray area and the white triangle.
2. The follower's problem—and therefore the bilevel problem—is infeasible for certain decisions of the leader, e.g., $x = 0$.
3. The set $\{(x, y) \colon x \in \Omega_x, y \in S(x)\}$ denotes the optimal follower solutions lifted to the x-y-space, and is given by the green and red facets.
4. This set is nonconvex!

An Academic and Linear Example (Kleinert 2021)

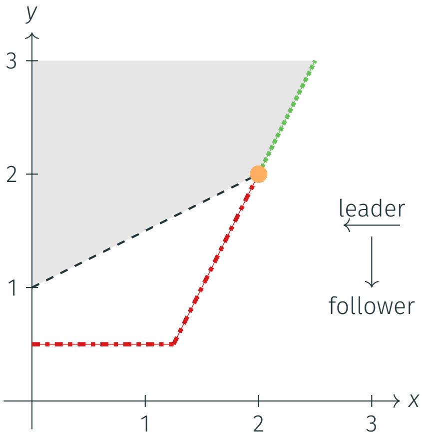


5. The single leader constraint (dashed line) renders certain optimal responses of the follower infeasible.
6. The bilevel feasible region $\mathcal{F}$ corresponds to the green facets.
7. Thus, the feasible set is not only nonconvex but also disconnected.
8. The optimal solution is $(3/7, 6/7)$ with objective function value $39/7$.
9. In contrast, ignoring the follower's objective, i.e., solving the single-level relaxation, yields the optimal solution $(3, 0)$ with objective function value 3. Note that the latter point is not bilevel feasible.

Independence of irrelevant constraints (Kleinert et al. 2021; Macal and Hurter 1997)

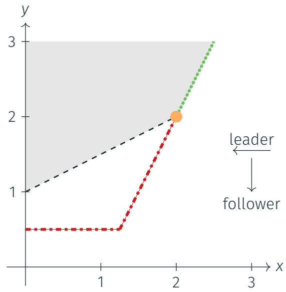
$$\begin{aligned} \min_{x,y \in \mathbb{R}} \quad & x \\ \text{s.t.} \quad & y \geq 0.5x + 1, \ x \geq 0 \\ & y \in \arg\min_{\bar{y} \in \mathbb{R}} \{\bar{y} : \bar{y} \geq 2x - 2, \ \bar{y} \geq 0.5\} \end{aligned}$$

Optimal solution: $(2, 2)$



Independence of irrelevant constraints (Kleinert et al. 2021; Macal and Hurter 1997)

- Strengthening $\bar{y} \geq 0.5$ in the lower-level problem using $y \geq 0.5x + 1$ of the upper-level problem
- This yields the minimum value of $0.5x + 1$ is 1 due to $x \geq 0$
- New bound of $\bar{y}$ is $\bar{y} \geq 1$
- Single-level relaxation stays the same



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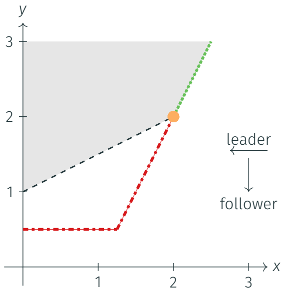
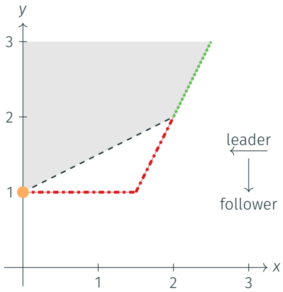
$$\begin{aligned} \min_{x,y \in \mathbb{R}} \quad & x \\ \text{s.t.} \quad & y \geq 0.5x + 1, \; x \geq 0, \\ & y \in \arg\min_{\bar{y} \in \mathbb{R}} \{\bar{y} \colon \bar{y} \geq 2x - 2, \; \bar{y} \geq 1\}, \end{aligned}$$

Optimal solution: $(0, 1) \neq (2, 2)$



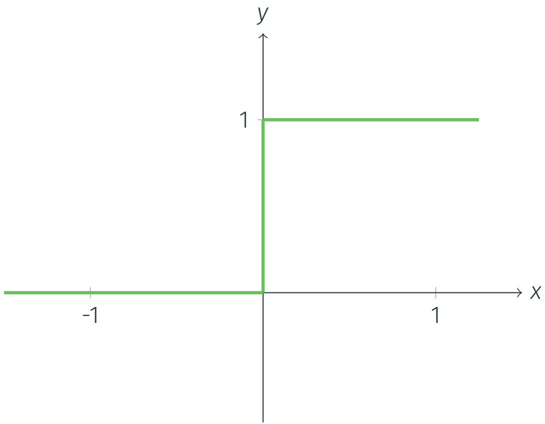


Optimistic vs. Pessimistic Bilevel Problems: Example 1.2 in Dempe et al. (2015)

$$\text{“}\min_x\text{”} \quad F(x,y) = x^2 + y \quad \text{s.t.} \quad y \in \mathcal{S}(x)$$

$$\mathcal{S}(x) = \arg\min_y \{-xy : 0 \leq y \leq 1\}$$

$$\mathcal{S}(x) = \begin{cases} [0,1], & x = 0, \\ \{0\}, & x < 0, \\ \{1\}, & x > 0. \end{cases}$$



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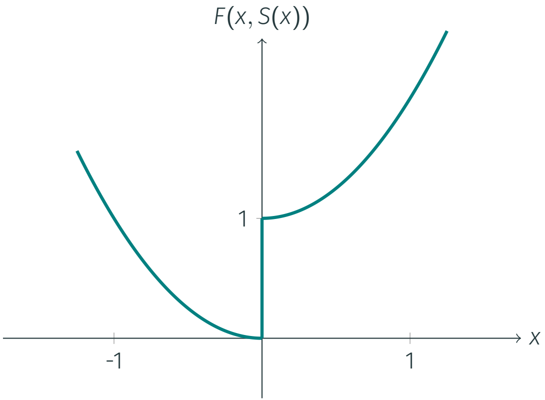

Pessimistic Bilevel Problem without Coupling Constraints

Definition

Consider the problem

$$\begin{aligned} \min_{x \in X} \quad & \max_{y \in \mathcal{S}(x)} F(x, y) \\ \text{s.t.} \quad & G(x) \geq 0, \\ & \mathcal{S}(x) \neq \emptyset, \end{aligned}$$

in which $\mathcal{S}(x)$ is the set of optimal solutions to the x-parameterized lower-level problem. Then, this problem is called the pessimistic bilevel problem without coupling constraints.

Pessimistic Bilevel Problem with Coupling Constraints

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The problem

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where $\mathcal{S}(x)$ is the set of optimal solutions to the x-parameterized lower-level problem is called the pessimistic bilevel problem with coupling constraints.

A Brief History of Complexity Results

- Jeroslow (1985)
  - Linear bilevel problems are NP-hard
- Ben-Ayed and Blair (1990)
  - Alternative proof using a reduction from KNAPSACK
- Hansen et al. (1992)
  - Linear bilevel problems are strongly NP-hard
  - Reduction from KERNEL
- Marcotte and Savard (2005)
  - "Easiest" proof
  - Special case of min-max problems without coupling constraints is strongly NP-hard
  - Reduction from 3-SAT
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  - Again via 3-SAT
  - As before: no coupling constraints are required

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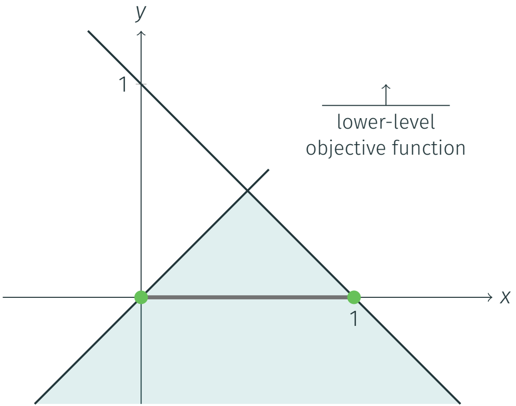
  - As before: no coupling constraints are required
- Completeness settled only recently: Buchheim (2023)

Why is it hard?

Audet et al. (1997): Modeling $x \in \{0, 1\}$ with LP-LP bilevel techniques

- additional lower-level variable y
- upper-level coupling constraint $y = 0$
- lower-level problem:

$$y = \arg\max_{\bar{y}} \{\bar{y} : \bar{y} \leq x, \bar{y} \leq 1 - x\}$$



Some theory of linear bilevel problems

The linear bilevel problem

We now consider LP-LP bilevel problems of the form

$$\begin{aligned} \min_{x,y} \quad & c_x^\top x + c_y^\top y \\ \text{s.t.} \quad & Ax + By \geq a, \\ & y \in \arg\min_{\bar{y}} \left\{ d^\top \bar{y} \colon Cx + D\bar{y} \geq b \right\} \end{aligned}$$

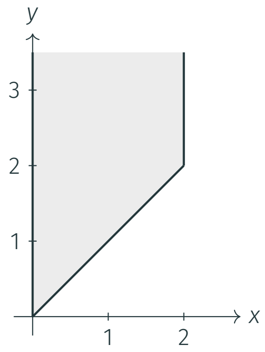
with $c_x \in \mathbb{R}^{n_x}$, $c_y, d \in \mathbb{R}^{n_y}$, $A \in \mathbb{R}^{m \times n_x}$, and $a \in \mathbb{R}^m$ as well as $C \in \mathbb{R}^{\ell \times n_x}$, $D \in \mathbb{R}^{\ell \times n_y}$, and $b \in \mathbb{R}^\ell$.

Unboundedness of the Shared Constraint Set

$$\begin{aligned}\max_{x,y} \quad & x + y \\ \text{s.t.} \quad & 0 \leq x \leq 2, \\ & y \in \arg\max_{\bar{y}} \{d\bar{y} : \bar{y} \geq x\}\end{aligned}$$

- Unbounded single-level relaxation

$$\max_{x,y} \quad x + y \quad \text{s.t.} \quad 0 \leq x \leq 2,\ y \geq x$$



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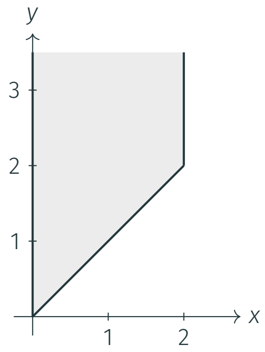

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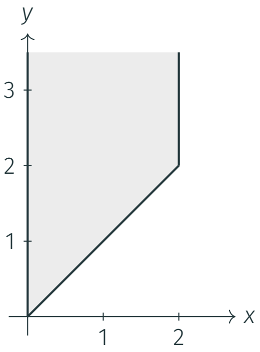
- $d > 0$: lower-level problem is unbounded for all decisions of the leader $\implies$ bilevel problem is infeasible



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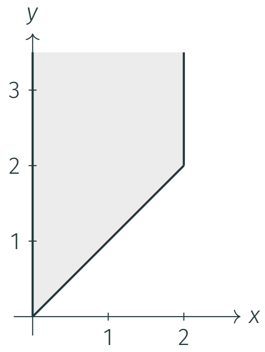
- $d = 0$: bilevel problem is unbounded
- $d > 0$: lower-level problem is unbounded for all decisions of the leader $\implies$ bilevel problem is infeasible
- $d < 0$: bilevel problem admits the unique optimal solution $(2, 2)$



Lemma (Lemma 2 of Xu and Wang (2014))

Suppose there exists a point $x \in \Omega_x$ for which the lower-level problem is unbounded. Then, the bilevel problemis infeasible.

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Theorem (See Theorem 1 in Fischetti et al. (2018))

Suppose that the shared constraint set Ω of Problem is non-empty. Moreover, let v^* be the optimal objective function value of the linear problem

$$\begin{aligned} \min_{\Delta y} \quad & d^\top \Delta y \\ \text{s.t.} \quad & D\Delta y \geq 0, \\ & -1 \leq \Delta y \leq 1. \end{aligned} \tag{1}$$

Then, for any $x \in \Omega_x$, the following is true. If $v^* < 0$ holds, the lower-level problem is unbounded. Otherwise, it has an optimal solution.

Existence of Solutions

Theorem

Suppose that the bilevel problem has a bounded shared constraint set Ω*. Then, the bilevel problem is either infeasible or admits an optimal solution.*

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Theorem

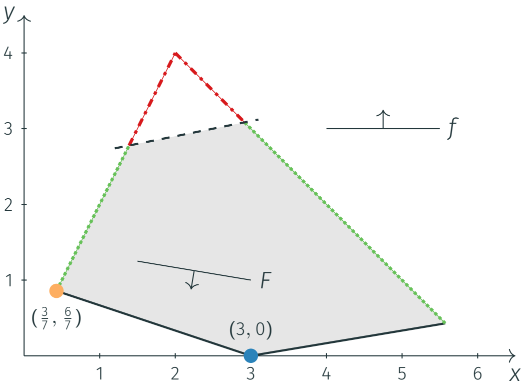
Suppose that the bilevel problem with $B = 0$ has a non-empty and bounded shared constraint set Ω. Then, the bilevel problem with $B = 0$ admits an optimal solution.

A first structural result

Theorem

Consider the LP-LP bilevel problem. Suppose that the shared constraint set Ω *is non-empty and bounded. The bilevel-feasible set can then be equivalently written as the intersection of the shared constraint set with the feasible points of a piecewise linear equality constraint. In particular, the bilevel-feasible set is a union of faces of the shared constraint set.*

The Academic Example Revisited



A first structural result: proof

We start by first re-writing the bilevel-feasible set

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and use the optimal-value function

$$\varphi(x) = \min_{y} \left\{d^\top y \colon Dy \geq b - Cx\right\}$$

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and use the optimal-value function

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again.

By using the strong-duality theorem, we can also express the optimal-value function by means of the dual LP as

$$\varphi(x) = \max_{\lambda} \left\{(b - Cx)^\top \lambda \colon D^\top \lambda = d,\ \lambda \geq 0\right\}.$$

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Let $\lambda^1, \ldots, \lambda^s$ be the set of all the dual polyhedron's vertices, i.e., the set of vertices of the polyhedron defined by

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Thus, we can further equivalently re-write the optimal-value function as

$$\varphi(x) = \max \left\{ (b - Cx)^\top \lambda \colon \lambda \in \{\lambda^1, \dots, \lambda^s\} \right\}.$$

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Thus, we can further equivalently re-write the optimal-value function as

$$\varphi(x) = \max \left\{ (b - Cx)^\top \lambda \colon \lambda \in \{\lambda^1, \dots, \lambda^s\} \right\}.$$

This shows that $\varphi(x)$ is a piecewise linear function and re-writing the bilevel-feasible set as

$$\mathcal{F} = \left\{ (x, y) \in \Omega \colon d^\top y - \varphi(x) = 0 \right\}$$

shows the claim that the bilevel-feasible set can be written as the intersection of the shared constraint set with a piecewise linear equality constraint.

A first structural result: proof

Consider now again the definition of the optimal-value function using the vertices of the dual polyhedron of the lower-level problem.

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$$0 = d^\top y - \varphi(x) = (D^\top \lambda^k)^\top y - (\lambda^k)^\top (b - Cx) = (\lambda^k)^\top (Cx + Dy - b).$$

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Thus, for those λ_i^k, $i \in \{1, \ldots, \ell\}$, with $\lambda_i^k > 0$ we get $(Cx + Dy - b)_i = 0$.

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In other words ...

Corollary

Suppose that the assumptions of the last theorem hold. Then, the LP-LP bilevel problem is equivalent to minimizing the upper-level objective function over the intersection of the shared constraint set with a piecewise-linear equality constraint.

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Corollary

Suppose that the assumptions of the last theorem hold and that the bilevel problem is feasible. Then, there always exists an optimal solution to the LP-LP bilevel problem that is a vertex of the bilevel-feasible set.

Solutions appear at vertices of the SLR

Theorem

Suppose that the assumptions of the last theorem hold and that the bilevel problem is feasible. Then, there always exists an optimal solution (x^*, y^*) *to the LP-LP bilevel problem that is a vertex of the shared constraint set* Ω.

Solutions appear at vertices of the SLR: proof

Let $(x^1, y^1), \ldots, (x^r, y^r)$ be the distinct vertices of the shared constraint set Ω.

Solutions appear at vertices of the SLR: proof

Let $(x^1, y^1), \dots, (x^r, y^r)$ be the distinct vertices of the shared constraint set Ω.

Since Ω is a convex polyhedron, any point in Ω can be written as a convex combination of these vertices, i.e.,

$$(x^*, y^*) = \sum_{i=1}^{r} \alpha_i (x^i, y^i)$$

with

$$\sum_{i=1}^{r} \alpha_i = 1 \quad \text{and} \quad \alpha_i \geq 0 \quad \text{for all } i = 1, \dots, r.$$

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From the proof of the last theorem it follows that the optimal-value function φ is convex and continuous.

Solutions appear at vertices of the SLR: proof

Since the bilevel solution (x^*, y^*) is, of course, bilevel feasible, we obtain

$$\begin{aligned}
0 &= d^\top y^* - \varphi(x^*) \\
&= d^\top \left(\sum_{i=1}^{r} \alpha_i y^i \right) - \varphi \left(\sum_{i=1}^{r} \alpha_i x^i \right) \\
&\geq \sum_{i=1}^{r} \alpha_i d^\top y^i - \sum_{i=1}^{r} \alpha_i \varphi(x^i) \\
&= \sum_{i=1}^{r} \alpha_i \left(d^\top y^i - \varphi(x^i) \right).
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&= \sum_{i=1}^{r} \alpha_i \left(d^\top y^i - \varphi(x^i)\right).
\end{aligned}
$$

By the definition of the optimal-value function we also have

$$
\varphi(x^i) = \min_y \left\{d^\top y \colon Cx^i + Dy \geq b\right\} \leq d^\top y^i.
$$

This implies $d^\top y^i - \varphi(x^i) \geq 0$.

Solutions appear at vertices of the SLR: proof

Consequently, for all $i \in \{1, \ldots, r\}$ with $\alpha_i > 0$ it holds $d^\top y^i = \varphi(x^i)$ since we otherwise get a contradiction on the last slide.

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From the last corollary we know that (x^*, y^*) is a vertex of the bilevel-feasible set. Suppose now that there are two indices i and j with $\alpha_i > 0$ and $\alpha_j > 0$.

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Thus, $(x^i, y^i) \in \mathcal{F}$ and $(x^j, y^j) \in \mathcal{F}$ holds and we can write (x^*, y^*) as a proper convex combination of two bilevel feasible points, which is a contradiction to the last corollary.

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Thus, (x^*, y^*) is a vertex of the shared constraint set.

How do you solve a linear bilevel problem?

Using optimality conditions

Most classic approach to obtain a single-level reformulation:

Exploit optimality conditions for the lower-level problem

Using optimality conditions

Most classic approach to obtain a single-level reformulation:

Exploit optimality conditions for the lower-level problem

- These optimality conditions need to be necessary and sufficient
- This is usually only possible for convex lower-level problems that satisfy a reasonable constraint qualification

An LP-LP Bilevel Problem

- Let's keep it simple: KKT reformulation of an LP-LP bilevel
- Consider

$$
\begin{aligned}
\min_{x,y} \quad & c_x^\top x + c_y^\top y \\
\text{s.t.} \quad & Ax + By \geq a, \\
& y \in \arg\min_{\bar{y}} \left\{ d^\top \bar{y} \colon Cx + D\bar{y} \geq b \right\}
\end{aligned}
$$

- Data: $c_x \in \mathbb{R}^{n_x}$, $c_y, d \in \mathbb{R}^{n_y}$, $A \in \mathbb{R}^{m \times n_x}$, $B \in \mathbb{R}^{m \times n_y}$, and $a \in \mathbb{R}^m$ as well as $C \in \mathbb{R}^{\ell \times n_x}$, $D \in \mathbb{R}^{\ell \times n_y}$, and $b \in \mathbb{R}^\ell$

KKT reformulation of LP-LP bilevel problems

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Lower-level problem can be seen as the x-parameterized linear problem

$$\min_{y} \quad d^\top y \quad \text{s.t.} \quad Dy \geq b - Cx$$

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$$

Lower-level problem can be seen as the x-parameterized linear problem

$$
\min_{y} \quad d^\top y \quad \text{s.t.} \quad Dy \geq b - Cx
$$

Its Lagrangian function is given by

$$
\mathcal{L}(y, \lambda) = d^\top y - \lambda^\top (Cx + Dy - b)
$$

KKT reformulation of LP-LP bilevel problems

The KKT conditions of the lower level are given by ...

- dual feasibility

$$D^\top \lambda = d, \quad \lambda \geq 0$$

- primal feasibility

$$Cx + Dy \geq b$$

- and the KKT complementarity conditions

$$\lambda_i (C_{i \cdot} x + D_{i \cdot} y - b_i) = 0 \quad \text{for all } i = 1, \dots, \ell$$

KKT reformulation of LP-LP bilevel problems

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- We now optimize over an extended space of variables including the lower-level dual variables λ
- Since we optimize over x, y, and λ simultaneously, any global solution of the problem above corresponds to an optimistic bilevel solution
- The KKT reformulation is linear except for the KKT complementarity conditions
- Thus, the problem is a nonconvex NLP

KKT reformulation of LP-LP bilevel problems

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& \lambda_i (C_{i\cdot} x + D_{i\cdot} y - b_i) = 0 \quad \text{for all } i = 1, \dots, \ell
\end{aligned}
$$

- ...
- Thus, the problem is a nonconvex NLP

It is even worse! It's a mathematical program with complementarity constraints (an MPCC).

KKT reformulation of LP-LP bilevel problems

$$
\begin{aligned}
\min_{x,y,\lambda} \quad & c_x^\top x + c_y^\top y \\
\text{s.t.} \quad & Ax + By \geq a, \; Cx + Dy \geq b \\
& D^\top \lambda = d, \; \lambda \geq 0 \\
& \lambda_i (C_{i\cdot} x + D_{i\cdot} y - b_i) = 0 \quad \text{for all } i = 1, \dots, \ell
\end{aligned}
$$

- ...
- Thus, the problem is a nonconvex NLP

It is even worse! It's a mathematical program with complementarity constraints (an MPCC).

Bad news (Ye and Zhu 1995)

Standard NLP algorithms usually cannot be applied for such problems since classic constraint qualifications like the Mangasarian–Fromowitz or the linear independence constraint qualification are violated at every feasible point.

Strong-Duality Based Reformulation

- Lower-level problem

$$\min_{y} \quad d^\top y \quad \text{s.t.} \quad Dy \geq b - Cx$$

- Dual lower-level problem

$$\max_{\lambda} \quad (b - Cx)^\top \lambda \quad \text{s.t.} \quad D^\top \lambda = d, \; \lambda \geq 0.$$

- Weak duality

$$d^\top y \geq (b - Cx)^\top \lambda$$

- Strong duality

$$d^\top y \leq (b - Cx)^\top \lambda$$

Strong-Duality Based Reformulation

$$
\begin{aligned}
\min_{x,y,\lambda} \quad & c_x^\top x + c_y^\top y \\
\text{s.t.} \quad & Ax + By \geq a, \; Cx + Dy \geq b \\
& D^\top \lambda = d, \; \lambda \geq 0 \\
& d^\top y \leq (b - Cx)^\top \lambda
\end{aligned}
$$

Back to the KKT reformulation: How to solve it?

Remember

The "only" reason for the nonconvexity of the KKT reformulation are the bilinear products of the lower-level dual variables λ_i and the upper-level primal variables x in the term

$$\lambda_i C_{i\cdot} x$$

Back to the KKT reformulation: How to solve it?

Remember

The "only" reason for the nonconvexity of the KKT reformulation are the bilinear products of the lower-level dual variables λ_i and the upper-level primal variables x in the term

$$\lambda_i C_{i\cdot} x$$

and the bilinear products of the lower-level dual variables λ_i and the lower-level primal variables y in the term

$$\lambda_i D_{i\cdot} y.$$

How to solve the KKT reformulation?

Key idea: Linearize these terms by exploiting the combinatorial structure of the KKT complementarity conditions.

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The complementarity conditions

$$\lambda_i(C_{i\cdot}x + D_{i\cdot}y - b_i) = 0, \quad i = 1, \ldots, \ell$$

can be seen as disjunctions stating that either

$$\lambda_i = 0 \quad \text{or} \quad C_{i\cdot}x + D_{i\cdot}y = b_i$$

needs to hold.

How to solve the KKT reformulation?

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can be seen as disjunctions stating that either

$$\lambda_i = 0 \quad \text{or} \quad C_{i\cdot}x + D_{i\cdot}y = b_i$$

needs to hold.

These two cases can be modeled using binary variables

$$z_i \in \{0, 1\}, \quad i = 1, \dots, \ell,$$

in the following mixed-integer linear way:

$$\lambda_i \leq M z_i, \quad C_{i\cdot}x + D_{i\cdot}y - b_i \leq M(1 - z_i).$$

Here, M is a sufficiently large constant.

How to solve the KKT reformulation?

By construction, we get the following result.

Theorem

Suppose that M is a sufficiently large constant. Then, the KKT reformulation is equivalent to the mixed-integer linear optimization problem

$$\begin{aligned}
\min_{x,y,\lambda,z} \quad & c_x^\top x + c_y^\top y \\
\text{s.t.} \quad & Ax + By \geq a, \ Cx + Dy \geq b, \\
& D^\top \lambda = d, \ \lambda \geq 0, \\
& \lambda_i \leq M z_i \quad \text{for all } i = 1, \ldots, \ell, \\
& C_{i\cdot} x + D_{i\cdot} y - b_i \leq M(1 - z_i) \quad \text{for all } i = 1, \ldots, \ell, \\
& z_i \in \{0, 1\} \quad \text{for all } i = 1, \ldots, \ell.
\end{aligned}$$

Be careful!

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Technical Note—There's No Free Lunch: On the Hardness of Choosing a Correct Big-M in Bilevel Optimization

Thomas Kleinert, Martine Labbé, Fr¨ank Plein, Martin Schmidt

Published Online: 30 Jun 2020 | https://doi.org/10.1287/opre.2019.1944

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Abstract

Abstract

One of the most frequently used approaches to solve linear bilevel optimization problems consists in replacing the lower-level problem with its Karush–Kuhn–Tucker (KKT) conditions and by reformulating the KKT complementarity conditions using techniques from mixed-integer linear optimization. The latter step requires to determine some big-*M* constant in order to bound the lower level's dual feasible set such that no bilevel-optimal solution is cut off. In practice, heuristics are often used to find a big-*M* although it is known that these approaches may fail. In this paper, we consider the hardness of two proxies for the above mentioned concept of a bilevel-correct big-*M*. First, we prove that verifying that a given big-*M* does not cut off any feasible vertex of the lower level's dual polyhedron cannot be done in polynomial time unless P = NP. Second, we show that verifying that a given big-*M* does not cut off any optimal point of the lower level's dual problem (for any point in the projection of the high-point relaxation onto the leader's decision space) is as hard as solving the original bilevel problem.

Complementarity-Based Branch-and-Bound

LP-LP bilevel problem

$$\begin{aligned} \min_{x,y} \quad & c_x^\top x + c_y^\top y \\ \text{s.t.} \quad & Ax + By \geq a \\ & y \in \arg\min_{\bar{y}} \left\{ d^\top \bar{y} : Cx + D\bar{y} \geq b \right\} \end{aligned}$$

Complementarity-Based Branch-and-Bound

LP-LP bilevel problem

$$\begin{aligned} \min_{x,y} \quad & c_x^\top x + c_y^\top y \\ \text{s.t.} \quad & Ax + By \geq a \\ & y \in \arg\min_{\bar{y}} \left\{ d^\top \bar{y} \colon Cx + D\bar{y} \geq b \right\} \end{aligned}$$

KKT reformulation as the MPCC

$$\begin{aligned} \min_{x,y,\lambda} \quad & c_x^\top x + c_y^\top y \\ \text{s.t.} \quad & Ax + By \geq a, \; Cx + Dy \geq b \\ & D^\top \lambda = d, \; \lambda \geq 0 \\ & \lambda_i (C_{i\cdot} x + D_{i\cdot} y - b_i) = 0 \quad \text{for all } i = 1, \dots, \ell \end{aligned}$$

- Start by solving the problem

$$\begin{aligned} \min_{x,y,\lambda} \quad & c_x^\top x + c_y^\top y \\ \text{s.t.} \quad & Ax + By \geq a, \; Cx + Dy \geq b, \\ & D^\top \lambda = d, \; \lambda \geq 0. \end{aligned}$$

  This is the single-level relaxation plus dual variables λ and the lower level's dual polyhedron given by

$$D^\top \lambda = d, \quad \lambda \geq 0$$

- Usually: $\exists \; i \in \{1, \dots, \ell\}$ with

$$\lambda_i (C_{i\cdot} x + D_{i\cdot} y - b_i) > 0$$

  holds.
- Construct two new sub-problems. one in which the primal constraint

$$C_{i\cdot} x + D_{i\cdot} y = b_i$$

  and one in which the dual fixing

$$\lambda_i = 0$$

Complementarity-Based Branch-and-Bound

$$
\begin{aligned}
\min_{x,y,\lambda} \quad & c_x^\top x + c_y^\top y \\
\text{s.t.} \quad & Ax + By \ge a, \; Cx + Dy \ge b \\
& D^\top \lambda = d, \; \lambda \ge 0 \\
& C_{i\cdot} x + D_{i\cdot} y = b_i \quad \text{for all } i \in \mathcal{P} \\
& \lambda_i = 0 \quad \text{for all } i \in \mathcal{D}
\end{aligned}
\tag{NP}
$$

Complementarity-Based Branch-and-Bound

1: Set $\mathcal{U} \leftarrow +\infty$ and $Q \leftarrow \{(\emptyset, \emptyset)\}$.
2: **while** $Q \neq \emptyset$ **do**
3: Choose any $(\mathcal{P}, \mathcal{D}) \in Q$ and set $Q \leftarrow Q \setminus \{(\mathcal{P}, \mathcal{D})\}$.
4: Solve Problem (NP) using $\mathcal{P}$ and $\mathcal{D}$.
5: **if** Problem (NP) for $\mathcal{P}$ and $\mathcal{D}$ is infeasible **then**
6: Go to Step 2.
7: **end if**
8: Let $(\bar{x}, \bar{y}, \bar{\lambda})$ denote the solution to Problem (NP) for $\mathcal{P}$ and $\mathcal{D}$.
9: **if** $c_x^\top \bar{x} + c_y^\top \bar{y} \geq \mathcal{U}$ **then**
10: Go to Step 2.
11: **end if**
12: **if** $(\bar{x}, \bar{y}, \bar{\lambda})$ satisfies $\bar{\lambda}_i(C_{i\cdot}\bar{x} + D_{i\cdot}\bar{y} - b_i) = 0$ for all $i \in \{1, \dots, \ell\}$ **then**
13: Set

$$(x^*, y^*, \lambda^*) \leftarrow (\bar{x}, \bar{y}, \bar{\lambda}), \quad \mathcal{U} \leftarrow c_x^\top x^* + c_y^\top y^*$$

and go to Step 2.
14: **end if**
15: Choose any $i \in \{1, \dots, \ell\}$ with $\bar{\lambda}_i(C_{i\cdot}\bar{x} + D_{i\cdot}\bar{y} - b_i) > 0$ and set

$$Q \leftarrow Q \cup \{(\mathcal{P} \cup \{i\}, \mathcal{D}), (\mathcal{P}, \mathcal{D} \cup \{i\})\}.$$

16: **end while**
17: **if** $\mathcal{U} < +\infty$ **then**
18: **return** (x^*, y^*, λ^*)
19: **else**
20: **return** the statement "The given LP-LP bilevel problem is infeasible."
21: **end if**

Theorem

Suppose that the root-node problem is bounded. Then, the complementarity-based branch-and-bound method terminates after a finite number of visited nodes with an optimal solution or with the correct indication of infeasibility.

About yesterday

I still owe you ...

- You have to copy $g(x, y) \geq 0$ for getting a correct single-level reformulation using the optimal-value function
  - Book: Exercise 1.14
- Even if both players optimize the same objective function but if there are coupling constraints, it is still a “proper” bilevel problem
  - Book: Example 1.21
- A tiny bit of history
  - Book: Section 1.3

Dempe and Dutta (2012)

Upper level

$$\begin{aligned} \min_{x,y} \quad & F(x,y) \\ \text{s.t.} \quad & G(x) \geq 0 \\ & y \in S(x) \end{aligned}$$

Lower level: x-parameterized convex problem

$$\begin{aligned} \min_{y} \quad & f(x,y) \\ \text{s.t.} \quad & g(x,y) \geq 0. \end{aligned}$$

- $y \mapsto f(x,y)$ is a smooth and convex function
- $y \mapsto g_i(x,y)$, $i = 1, \ldots, \ell$, are smooth and concave functions for all x with $G(x) \geq 0$

Dempe and Dutta (2012) … continued

Single-level KKT reformulation

$$
\begin{aligned}
\min_{x,y,\lambda} \quad & F(x,y) \\
\text{s.t.} \quad & G(x) \geq 0, \\
& \nabla_y \mathcal{L}(x,y,\lambda) = 0, \\
& g(x,y) \geq 0, \\
& \lambda \geq 0, \\
& \lambda^\top g(x,y) = 0.
\end{aligned}
$$

Theorem (Theorem 2.1 in Dempe and Dutta (2012))

Let (x^*, y^*) *be a globally optimal solution to the latter bilevel problem and assume that the lower-level problem is a convex optimization problem that satisfies Slater's constraint qualification at* x^**. Then, the point* (x^*, y^*, λ^*) *is a globally optimal solution to the single-level KKT reformulation for every*

$$\lambda^* \in \Lambda(x^*, y^*) := \left\{ \lambda \in \mathbb{R}^\ell \colon \nabla_y \mathcal{L}(x^*, y^*, \lambda) = 0, \; \lambda \geq 0, \; \lambda^\top g(x^*, y^*) = 0 \right\}.$$

Dempe and Dutta (2012) … continued

Theorem (Theorem 2.3 in Dempe and Dutta (2012))

Let (x^*, y^*, λ^*) *be a globally optimal solution to the single-level KKT reformulation and let the lower-level problem be convex. Moreover, suppose that for all* x *with* $G(x) \geq 0$*, the lower-level problem satisfies Slater's constraint qualification. Then,* (x^*, y^*) *is a globally optimal solution to the bilevel problem.*

Dempe and Dutta (2012) ... continued

Theorem (Theorem 2.3 in Dempe and Dutta (2012))

Let (x^*, y^*, λ^*) be a globally optimal solution to the single-level KKT reformulation and let the lower-level problem be convex. Moreover, suppose that for all x with $G(x) \geq 0$, the lower-level problem satisfies Slater's constraint qualification. Then, (x^*, y^*) is a globally optimal solution to the bilevel problem.

However: The local minimizers of the KKT reformulation do not need to correspond tolocal optima ofthebilevel problem!

Book: Section 3.6

The burial of coupling constraints in linear bilevel optimization

Linear Bilevel Optimization

$$\begin{aligned} \min_{x \in X, y} \quad & c^\top x + d^\top y \\ \text{s.t.} \quad & Ax + By \geq a \\ & y \in S(x) \end{aligned}$$

$S(x)$: set of optimal solutions to the x-parameterized lower-level problem

$$\min_{y} \quad f^\top y \quad \text{s.t.} \quad Cx + Dy \geq b$$

Linear Bilevel Optimization

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$S(x)$: set of optimal solutions to the x-parameterized lower-level problem

$$\min_y \quad f^\top y \quad \text{s.t.} \quad Cx + Dy \geq b$$

Assumptions

- $X \subseteq \mathbb{R}^n$ is a polyhedron
- All vectors and matrices have rational entries
- Bilevel problem is solvable

Linear Bilevel Optimization: Coupling Constraints

$$\begin{aligned} \min_{x \in X, y} \quad & c^\top x + d^\top y \\ \text{s.t.} \quad & Ax + By \geq a \\ & y \in S(x) \end{aligned}$$

$S(x)$: set of optimal solutions to the x-parameterized lower-level problem

$$\min_{y} \quad f^\top y \quad \text{s.t.} \quad Cx + Dy \geq b$$

Linear Bilevel Optimization: Coupling Constraints

$$
\begin{aligned}
\min_{x \in X, y} \quad & c^\top x + d^\top y \\
\text{s.t.} \quad & Ax + By \geq a \\
& y \in S(x)
\end{aligned}
$$

$S(x)$: set of optimal solutions to the x-parameterized lower-level problem

$$
\min_{y} \quad f^\top y \quad \text{s.t.} \quad Cx + Dy \geq b
$$

Does $B = 0$ make a difference?

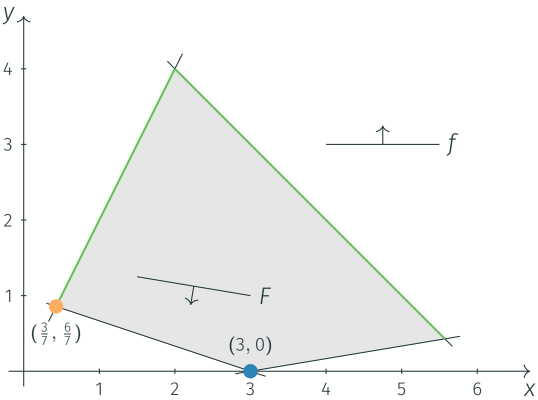
An Academic Example (Kleinert 2021): No Coupling Constraints

Upper level

$$\min_{x,y} \quad F(x,y) = x + 6y$$
$$\text{s.t.} \quad x \geq 0, \; y \in S(x)$$

Lower level

$$\min_{y} \quad f(x,y) = -y$$
$$\text{s.t.} \quad 2x - y \geq 0, \; -x - y \geq -6$$
$$-x + 6y \geq -3, \; x + 3y \geq 3$$



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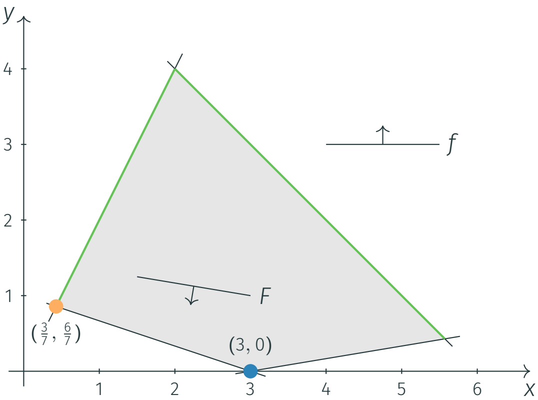


- Both levels are linear optimization problems and all variables are continuous
- Polyhedral shared constraint set
- Nonconvex feasible set

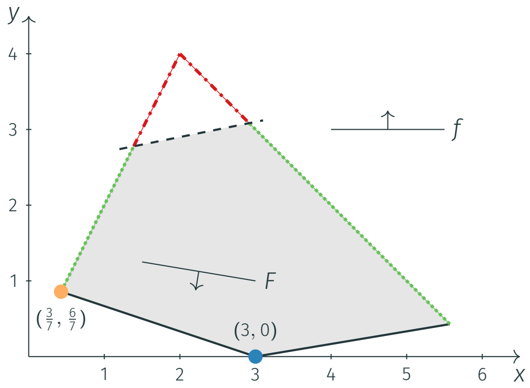
An Academic Example (Kleinert 2021): With a Coupling Constraint

Upper level

$$
\begin{aligned}
\min_{x,y} \quad & F(x,y) = x + 6y \\
\text{s.t.} \quad & -x + 5y \le 12.5 \\
& x \ge 0, \; y \in S(x)
\end{aligned}
$$

Lower level

$$
\begin{aligned}
\min_{y} \quad & f(x,y) = -y \\
\text{s.t.} \quad & 2x - y \ge 0, \; -x - y \ge -6 \\
& -x + 6y \ge -3, \; x + 3y \ge 3
\end{aligned}
$$



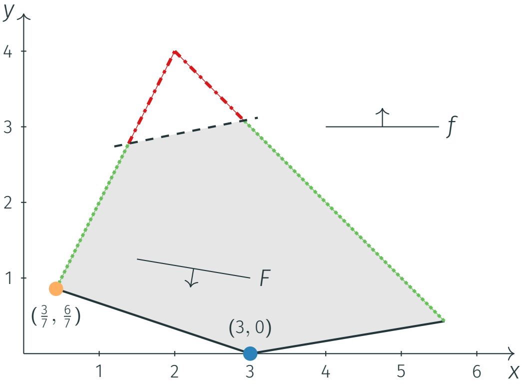
An Academic Example (Kleinert 2021): With a Coupling Constraint

Upper level

$$\begin{aligned} \min_{x,y} \quad & F(x,y) = x + 6y \\ \text{s.t.} \quad & -x + 5y \leq 12.5 \\ & x \geq 0, \; y \in S(x) \end{aligned}$$

Lower level

$$\begin{aligned} \min_{y} \quad & f(x,y) = -y \\ \text{s.t.} \quad & 2x - y \geq 0, \; -x - y \geq -6 \\ & -x + 6y \geq -3, \; x + 3y \geq 3 \end{aligned}$$



- Both levels are linear optimization problems and all variables are continuous
- Polyhedral shared constraint set
- Nonconvex feasible set
- Feasible set is even disconnected

The Impact of Coupling Constraints: (Dis)connectedness

- Coupling constraints may lead to disconnected feasible sets
- Benson (1989): the feasible set of a linear bilevel problem with $B = 0$ is always connected

The Impact of Coupling Constraints: (Dis)connectedness

- Coupling constraints may lead to disconnected feasible sets
- Benson (1989): the feasible set of a linear bilevel problem with $B = 0$ is always connected

More modeling power?

Harder to solve?

What About Complexity Theory?

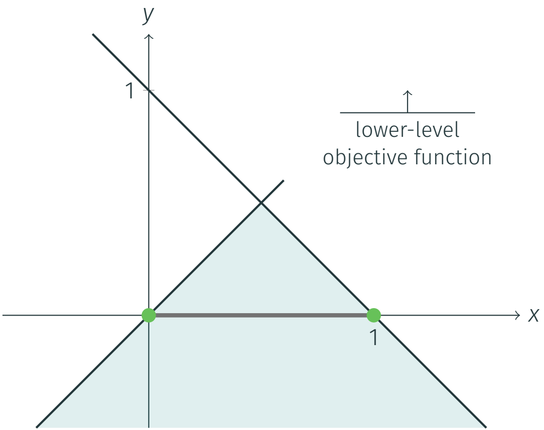
Audet et al. (1997): $x \in \{0, 1\}$ can be modeled in purely continuous linear bilevel models.

What About Complexity Theory?

Audet et al. (1997): $x \in \{0, 1\}$ can be modeled in purely continuous linear bilevel models.

- Additional lower-level variable y
- Upper-level constraints $y = 0$
- Lower-level problem

$$y = \arg\max_{\bar{y}} \{\bar{y} : \bar{y} \leq x,\ \bar{y} \leq 1 - x\}$$



What About Complexity Theory?

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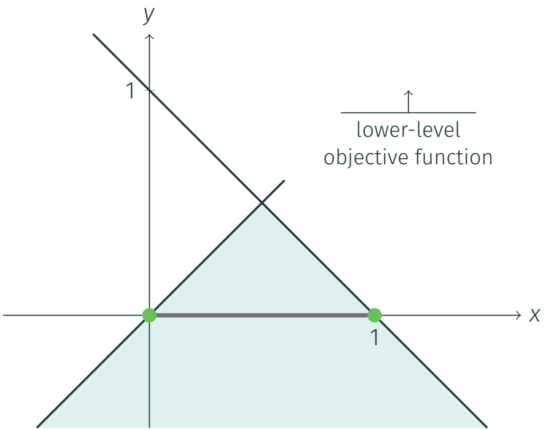


Linear mixed-binary optimization is a special case of continuous linear bilevel optimization

A Brief History About Complexity

- Jeroslow (1985)
    - Linear bilevel problems are NP-hard
- Ben-Ayed and Blair (1990)
    - Alternative proof using a reduction from KNAPSACK
- Hansen et al. (1992)
    - Linear bilevel problems are strongly NP-hard
    - Reduction from KERNEL
- Marcotte and Savard (2005)
    - “Easiest” proof
    - Special case of min-max problems without coupling constraints is strongly NP-hard
    - Reduction from 3-SAT
- Vicente et al. (1994)
    - Checking local optimality of a given point is strongly NP-hard
    - Again via 3-SAT
    - As before: no coupling constraints are required
- Completeness settled only recently: Buchheim (2023)

Hmm ...

- Coupling constraints allow for modeling a richer class of feasible sets
- Stronger “modeling capabilities” do not result in any change of the hardness

Hmm ...

- Coupling constraints allow for modeling a richer class of feasible sets
- Stronger “modeling capabilities” do not result in any change of the hardness

Research Questions

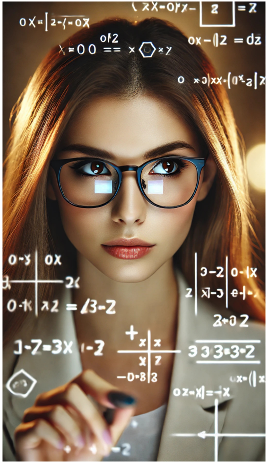
- What about optimal solutions?
- Do we “really” increase modeling capabilities by using coupling constraints?

2024: The Optimistic Case

The Team

Henri Lefebvre

Dorothee Henke



Johannes Thürauf

Do we “really” increase modeling capabilities by using coupling constraints?

Spoiler: No!

Do we “really” increase modeling capabilities by using coupling constraints?

Spoiler: No!

Why not?

For every given linear bilevel optimization problem with coupling constraints, we derive ...

- a linear bilevel problem without coupling constraints
- that has the same set of optimal solutions

Here are all the details ...

On coupling constraints in linear bilevel optimization

Short Communication | Open access | Published: 03 December 2024

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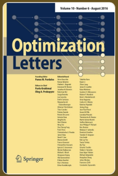

Dorothee Henke, Henri Lefebvre, Martin Schmidt & Johannes Thürauf

Abstract

It is well-known that coupling constraints in linear bilevel optimization can lead to disconnected feasible sets, which is not possible without coupling constraints. However, there is no difference between linear bilevel problems with and without coupling constraints w.r.t. their complexity-theoretical hardness. In this note, we prove that, although there is a clear difference between these two classes of problems in terms of their feasible sets, the classes are equivalent on the level of optimal solutions. To this end, given

Re-Writing the Problem

Upper level

$$
\begin{aligned}
\min_{x,y,\varepsilon} \quad & c^\top x + d^\top y \\
\text{s.t.} \quad & x \in X \\
& \varepsilon = 0 \\
& (y, \varepsilon) \in \tilde{S}(x)
\end{aligned}
\tag{R}
$$

Lower level

$$
\begin{aligned}
\min_{y,\varepsilon} \quad & f^\top y \\
\text{s.t.} \quad & Ax + By + \varepsilon e \geq a \\
& Cx + Dy \geq b \\
& \varepsilon \geq 0
\end{aligned}
$$

Re-Writing the Problem

Upper level

$$\begin{aligned} \min_{x,y,\varepsilon} \quad & c^\top x + d^\top y \\ \text{s.t.} \quad & x \in X \\ & \varepsilon = 0 \\ & (y, \varepsilon) \in \tilde{S}(x) \end{aligned} \tag{R}$$

Lower level

$$\begin{aligned} \min_{y,\varepsilon} \quad & f^\top y \\ \text{s.t.} \quad & Ax + By + \varepsilon e \geq a \\ & Cx + Dy \geq b \\ & \varepsilon \geq 0 \end{aligned}$$

Lemma

For every bilevel feasible point (x, y) of the original bilevel problem, the point $(x, y, 0)$ is bilevel feasible for Problem (R) with the same objective value. For every bilevel feasible point (x, y, ε) of Problem (R), the point (x, y) is bilevel feasible for the original bilevel problem with the same objective value.

Re-Writing the Problem & Penalize

Theorem

There is a poly-sized parameter $\kappa > 0$ (in the bit-encoding length of the problem's data) so that the bilevel problem (without coupling constraints)

$$\begin{aligned} \min_{x,y,\varepsilon} \quad & c^\top x + d^\top y + \kappa\varepsilon \\ \text{s.t.} \quad & x \in X,\ (y,\varepsilon) \in \tilde{S}(x) \end{aligned} \tag{P}$$

has the same set of optimal solutions as Problem (R)*.*

Re-Writing the Problem & Penalize

Theorem

There is a poly-sized parameter $\kappa > 0$ (in the bit-encoding length of the problem's data) so that the bilevel problem (without coupling constraints)

$$\begin{aligned} \min_{x,y,\varepsilon} \quad & c^\top x + d^\top y + \kappa\varepsilon \\ \text{s.t.} \quad & x \in X, \ (y, \varepsilon) \in \tilde{S}(x) \end{aligned} \tag{P}$$

has the same set of optimal solutions as Problem (R).

That's surprising!

- Reason #1
  - Feasible region of the original problem is nonconvex and disconnected
  - Ye and Zhu (1995): no constraint qualification is satisfied
  - Exact penalization usually fails!

Re-Writing the Problem & Penalize

Theorem

There is a poly-sized parameter $\kappa > 0$ (in the bit-encoding length of the problem's data) so that the bilevel problem (without coupling constraints)

$$\begin{aligned} \min_{x,y,\varepsilon} \quad & c^\top x + d^\top y + \kappa\varepsilon \\ \text{s.t.} \quad & x \in X, \ (y,\varepsilon) \in \tilde{S}(x) \end{aligned} \tag{P}$$

has the same set of optimal solutions as Problem (R).

That's surprising!

- Reason #1
    - Feasible region of the original problem is nonconvex and disconnected
    - Ye and Zhu (1995): no constraint qualification is satisfied
    - Exact penalization usually fails!
- Reason #2
    - Exact penalty functions are usually nonsmooth (à la ℓ_1)
    - Our penalty function is perfectly smooth (even linear)

Proof Idea

1. We derive a single-level reformulation of the bilevel problem (R), using the KKT conditions of the follower's problem.
2. We apply results from augmented Lagrangian duality theory for mixed-integer linear problems to show that a poly-sized exact penalization parameter exists.
3. We show that the resulting mixed-integer linear program is nothing but the KKT reformulation of Problem (P).

Proof

- Lower-level problem of Problem (R) is an LP
- Dempe and Dutta (2012): Replace it with its KKT conditions

$$
\begin{aligned}
\min_{x,y,\varepsilon} \quad & c^\top x + d^\top y \\
\text{s.t.} \quad & x \in X, \ \varepsilon = 0 \\
& Ax + By + \varepsilon e \geq a, \ Cx + Dy \geq b, \ \varepsilon \geq 0 \\
& B^\top \lambda + D^\top \mu = f, \ e^\top \lambda + \eta = 0 \\
& \lambda, \mu, \eta \geq 0, \\
& \lambda^\top (Ax + By + \varepsilon e - a) = 0, \ \mu^\top (Cx + Dy - b) = 0, \ \eta\varepsilon = 0
\end{aligned}
$$

- Additional binary variables z^λ, z^μ, z^η
- Sufficiently large big-M

$$
\begin{aligned}
& \lambda \leq (1 - z^\lambda)M, \quad \mu \leq (1 - z^\mu)M, \quad \eta \leq (1 - z^\eta)M \\
& Ax + By + \varepsilon e - a \leq z^\lambda M, \quad Cx + Dy - b \leq z^\mu M, \quad \varepsilon \leq z^\eta M
\end{aligned}
$$

Wait! Are we cheating?

- Pineda and Morales (2019): Heuristics for computing big-M values usually fail
- Kleinert et al. (2020): Validating the correctness of a given big-M is as hard as the original bilevel problem

Wait! Are we cheating?

- Pineda and Morales (2019): Heuristics for computing big-M values usually fail
- Kleinert et al. (2020): Validating the correctness of a given big-M is as hard as the original bilevel problem

But ...

- Buchheim (2023): valid and poly-sized M can be computed in polynomial time

Proof … continued

We have the MILP

$$
\begin{aligned}
\min_{x,y,\varepsilon,z^\lambda,z^\mu,z^\eta} \quad & c^\top x + d^\top y \\
\text{s.t.} \quad & x \in X,\ \varepsilon = 0 \\
& Ax + By + \varepsilon e \geq a,\ Cx + Dy \geq b,\ \varepsilon \geq 0 \\
& B^\top \lambda + D^\top \mu = f,\ e^\top \lambda + \eta = 0,\ \lambda, \mu, \eta \geq 0 \\
& \lambda \leq (1 - z^\lambda)M,\ \mu \leq (1 - z^\mu)M,\ \eta \leq (1 - z^\eta)M \\
& Ax + By + \varepsilon e - a \leq z^\lambda M,\ Cx + Dy - b \leq z^\mu M,\ \varepsilon \leq z^\eta M
\end{aligned}
$$

Proof ... continued

We have the MILP

$$
\begin{aligned}
\min_{x,y,\varepsilon,z^\lambda,z^\mu,z^\eta} \quad & c^\top x + d^\top y \\
\text{s.t.} \quad & x \in X, \ \varepsilon = 0 \\
& Ax + By + \varepsilon e \geq a, \ Cx + Dy \geq b, \ \varepsilon \geq 0 \\
& B^\top \lambda + D^\top \mu = f, \ e^\top \lambda + \eta = 0, \ \lambda, \mu, \eta \geq 0 \\
& \lambda \leq (1 - z^\lambda)M, \ \mu \leq (1 - z^\mu)M, \ \eta \leq (1 - z^\eta)M \\
& Ax + By + \varepsilon e - a \leq z^\lambda M, \ Cx + Dy - b \leq z^\mu M, \ \varepsilon \leq z^\eta M
\end{aligned}
$$

ℓ_∞ penalization of the coupling constraint $\varepsilon = 0$

$$
\begin{aligned}
\min_{x,y,\varepsilon} \quad & c^\top x + d^\top y + \kappa \varepsilon \\
\text{s.t.} \quad & \text{all constraints except from } \varepsilon = 0
\end{aligned}
$$

Proof ... continued: What about κ?

Home > Mathematical Programming > Article

Exact augmented Lagrangian duality for mixed integer linear programming

Full Length Paper | Series A | Published: 21 April 2016

Volume 161, pages 365–387, (2017) Cite this article

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Mohammad Javad Feizollahi, Shabbir Ahmed & Andy Sun

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Abstract

We investigate the augmented Lagrangian dual (ALD) for mixed integer linear programming (MIP) problems. ALD modifies the classical Lagrangian dual by appending a nonlinear penalty function on the violation of the dualized constraints in order to reduce the duality gap. We first provide a primal characterization for ALD for MIPs and prove that ALD is able to asymptotically achieve zero duality gap when the weight on the penalty function is allowed to go to infinity. This provides an alternative characterization and proof of a recent result in Boland and Eberhard (Math Program 150(2):491–509, 2015, Proposition 3). We further show that, under some mild conditions, ALD using any norm as

Feizollahi et al. (2016)

- Theorem 4: duality gap for the augmented Lagrangian dual of a solvable (mixed-integer) linear optimization problem can be closed by using a norm as the augmenting function and a sufficiently large but finite penalty parameter.
- Proposition 1: Optimal solutions of MILP reformulation and the ℓ_∞ penalty problem are the same

Gu et al. (2020)

- Theorem 22: Penalty parameter can be chosen to be of polynomial size in case of the ℓ_∞-norm

Proof … continued: back to the slide before

We have the MILP

$$
\begin{aligned}
\min_{x,y,\varepsilon,z^\lambda,z^\mu,z^\eta} \quad & c^\top x + d^\top y \\
\text{s.t.} \quad & x \in X, \; \varepsilon = 0, \\
& Ax + By + \varepsilon e \geq a, \; Cx + Dy \geq b, \; \varepsilon \geq 0 \\
& B^\top \lambda + D^\top \mu = f, \; e^\top \lambda + \eta = 0, \; \lambda, \mu, \eta \geq 0 \\
& \lambda \leq (1 - z^\lambda)M, \; \mu \leq (1 - z^\mu)M, \; \eta \leq (1 - z^\eta)M \\
& Ax + By + \varepsilon e - a \leq z^\lambda M, \; Cx + Dy - b \leq z^\mu M, \; \varepsilon \leq z^\eta M
\end{aligned}
$$

ℓ_∞ penalization of the coupling constraint $\varepsilon = 0$

$$
\begin{aligned}
\min_{x,y,\varepsilon} \quad & c^\top x + d^\top y + \kappa\varepsilon \\
\text{s.t.} \quad & \text{all constraints except from } \varepsilon = 0
\end{aligned}
$$

This is the KKT reformulation of the bilevel problem from the theorem!

κ ... One more time!

Feizollahi et al. (2016) & Gu et al. (2020)

Existence of finite and poly-sized
exact penalty parameter.

Open (until 2024)

Can it be computed in polynomial time?

Feizollahi et al. (2016) & Gu et al. (2020)

Existence of finite and poly-sized exact penalty parameter.

Open (until 2024)

Can it be computed in polynomial time?

Yes! Lemma 4 of Lefebvre and Schmidt (2024)

Exact Augmented Lagrangian Duality for Nonconvex Mixed-Integer Nonlinear Optimization

Henri Lefebvre, Martin Schmidt

Abstract. In the context of mixed-integer nonlinear problems (MINLPs), it is well-known that strong duality does not hold in general if the standard Lagrangian dual is used. Hence, we consider the augmented Lagrangian dual (ALD), which adds a nonlinear penalty function to the classic Lagrangian function. For this setup, we study conditions under which the ALD leads to a zero duality gap for general MINLPs. In particular, under mild assumptions and for a large class of penalty functions, we show that the ALD yields zero duality gaps if the penalty parameter goes to infinity. If the penalty function is a norm, we also show that the ALD leads to zero duality gaps for a finite penalty parameter. Moreover, we show that such a finite penalty parameter can be computed in polynomial time in the mixed-integer linear case. This generalizes the recent results on linearly constrained mixed-integer problems by Bhardwaj et al. (2024), Boland and Eberhard (2014), Feizollahi et al. (2016), and Gu et al. (2020).

2025: The Pessimistic Case

Optimistic bilevel problem without coupling constraints

$$\min_{x \in X} \quad F_o(x) := c^\top x + \min_y \left\{ d^\top y : y \in S(x) \right\}$$

$S(x)$: set of optimal solutions to the x-parameterized optimization problem

$$\min_y \quad f^\top y \quad \text{s.t.} \quad Cx + Dy \geq b$$

Optimistic Bilevel Problems with and without Coupling Constraints

Optimistic bilevel problem without coupling constraints

$$\min_{x \in X} \quad F_{\mathrm{o}}(x) := c^\top x + \min_{y} \left\{ d^\top y \colon y \in S(x) \right\}$$

$S(x)$: set of optimal solutions to the x-parameterized optimization problem

$$\min_{y} \quad f^\top y \quad \text{s.t.} \quad Cx + Dy \geq b$$

Optimistic bilevel problem with coupling constraints

$$\min_{x \in X} \quad F_{\mathrm{oc}}(x) := c^\top x + \min_{y} \left\{ d^\top y \colon y \in S(x),\ Ax + By \geq a \right\}$$

Pessimistic Bilevel Problems with and without Coupling Constraints

Pessimistic bilevel problem without coupling constraints

$$\min_{x \in \bar{X}} \quad F_{\mathrm{p}}(x) := c^\top x + \max_{y} \left\{ d^\top y \colon y \in S(x) \right\}$$

with

$$\bar{X} := X \cap \{x \colon S(x) \neq \emptyset\}$$

Pessimistic Bilevel Problems with and without Coupling Constraints

Pessimistic bilevel problem without coupling constraints

$$\min_{x \in \bar{X}} \quad F_{\mathrm{p}}(x) := c^\top x + \max_{y} \left\{ d^\top y \colon y \in S(x) \right\}$$

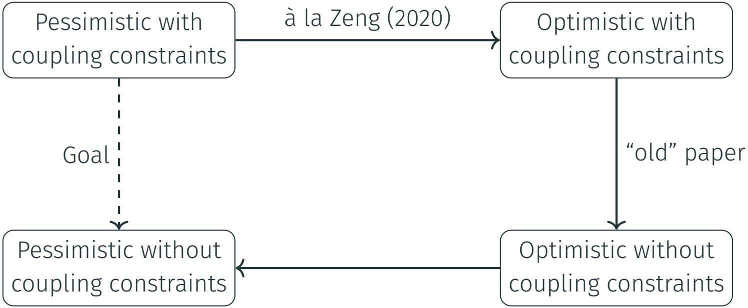
with

$$\bar{X} := X \cap \{x \colon S(x) \neq \emptyset\}$$

Pessimistic bilevel problem with coupling constraints

$$\begin{aligned} \min_{x \in \bar{X}} \quad & F_{\mathrm{pc}}(x) := c^\top x \\ \text{s.t.} \quad & Ax + By \geq a \quad \text{for all } y \in S(x) \end{aligned}$$

Bye bye, coupling constraints ...



The pessimistic coupling constraint

$$Ax + By \geq a \quad \text{for all } y \in S(x)$$

is equivalent to

$$A_{i\cdot}x + B_{i\cdot}y \geq a_i \quad \text{for all } y \in S(x) \text{ and all } i \in [m] := \{1, \dots, m\}\,.$$

The pessimistic coupling constraint

$$Ax + By \geq a \quad \text{for all } y \in S(x)$$

is equivalent to

$$A_{i\cdot}x + B_{i\cdot}y \geq a_i \quad \text{for all } y \in S(x) \text{ and all } i \in [m] := \{1, \dots, m\}.$$

Lemma (à la Zeng (2020))

Let $x \in X$ be given and consider a fixed $i \in [m]$. Then, x satisfies the i-th coupling constraint if and only if there exist $\bar{y}$ and

$$y^i \in \arg\min \left\{ B_{i\cdot}y \colon Dy \geq b - Cx,\ f^\top y \leq f^\top \bar{y} \right\}$$

that satisfy

$$D\bar{y} \geq b - Cx, \quad B_{i\cdot}y^i \geq a_i - A_{i\cdot}x.$$

Proof of the “à la Zeng (2020)” lemma

Let $x \in X$ be given and consider a fixed $i \in [m]$. Then, the i-th coupling constraint is equivalent to $\min_y\{B_{i\cdot}y : y \in S(x)\} \geq a_i - A_{i\cdot}x$, which can be reformulated as $B_{i\cdot}y^i \geq a_i - A_{i\cdot}x$ with $y^i \in \arg\min_y \{B_{i\cdot}y : y \in S(x)\}$.

Proof of the “à la Zeng (2020)” lemma

Let $x \in X$ be given and consider a fixed $i \in [m]$. Then, the i-th coupling constraint is equivalent to $\min_y\{B_{i\cdot}y \colon y \in S(x)\} \geq a_i - A_{i\cdot}x$, which can be reformulated as $B_{i\cdot}y^i \geq a_i - A_{i\cdot}x$ with $y^i \in \arg\min_y \{B_{i\cdot}y \colon y \in S(x)\}$. Now, let φ denote the optimal-value function of the lower-level problem. It follows that x satisfies the i-th coupling constraint if and only if

$$B_{i\cdot}y^i \geq a_i - A_{i\cdot}x \quad \text{with} \quad y^i \in \arg\min_y \left\{B_{i\cdot}y \colon Dy \geq b - Cx,\ f^\top y \leq \varphi(x)\right\}.$$

Proof of the “à la Zeng (2020)” lemma

Let $x \in X$ be given and consider a fixed $i \in [m]$. Then, the i-th coupling constraint is equivalent to $\min_y\{B_{i\cdot}y : y \in S(x)\} \geq a_i - A_{i\cdot}x$, which can be reformulated as $B_{i\cdot}y^i \geq a_i - A_{i\cdot}x$ with $y^i \in \arg\min_y \{B_{i\cdot}y : y \in S(x)\}$. Now, let φ denote the optimal-value function of the lower-level problem. It follows that x satisfies the i-th coupling constraint if and only if

$$B_{i\cdot}y^i \geq a_i - A_{i\cdot}x \quad \text{with} \quad y^i \in \arg\min_y \left\{B_{i\cdot}y : Dy \geq b - Cx,\ f^\top y \leq \varphi(x)\right\}.$$

We now show that the latter is equivalent to the stated conditions in the lemma.

First, let us assume that the above holds. Then, there exists $\bar{y}$ such that $\varphi(x) = f^\top \bar{y}$ and $D\bar{y} \geq b - Cx$ is satisfied. Hence, the conditions of the lemma hold.

Proof of the "à la Zeng (2020)" lemma

Let $x \in X$ be given and consider a fixed $i \in [m]$. Then, the i-th coupling constraint is equivalent to $\min_y \{B_{i\cdot}y \colon y \in S(x)\} \geq a_i - A_{i\cdot}x$, which can be reformulated as $B_{i\cdot}y^i \geq a_i - A_{i\cdot}x$ with $y^i \in \arg\min_y \{B_{i\cdot}y \colon y \in S(x)\}$. Now, let φ denote the optimal-value function of the lower-level problem. It follows that x satisfies the i-th coupling constraint if and only if

$$B_{i\cdot}y^i \geq a_i - A_{i\cdot}x \quad \text{with} \quad y^i \in \arg\min_y \left\{B_{i\cdot}y \colon Dy \geq b - Cx,\ f^\top y \leq \varphi(x)\right\}.$$

We now show that the latter is equivalent to the stated conditions in the lemma.

First, let us assume that the above holds. Then, there exists $\bar{y}$ such that $\varphi(x) = f^\top \bar{y}$ and $D\bar{y} \geq b - Cx$ is satisfied. Hence, the conditions of the lemma hold.

Conversely, assume that the conditions of the lemma are satisfied. The feasibility of $\bar{y}$ implies

$$\min_y \left\{B_{i\cdot}y \colon Dy \geq b - Cx,\ f^\top y \leq f^\top \bar{y}\right\} \leq \min_y \left\{B_{i\cdot}y \colon Dy \geq b - Cx,\ f^\top y \leq \varphi(x)\right\}.$$

This concludes the proof.

From Pessimistic to Optimistic Bilevel Optimization with Coupling Constraints

Theorem

Let $\mathcal{S}$ be the set of globally optimal solutions to the pessimistic bilevel problem with coupling constraints. Moreover, let $\tilde{\mathcal{S}}$ be the set of globally optimal solutions to the optimistic single-leader multi-follower problem

$$\min_{(x,\bar{y})\in\tilde{X}} \; c^\top x + \min_{y} \left\{0 \colon y^i \in \tilde{S}^i(x,\bar{y}),\; B_{i\cdot}y^i \geq a_i - A_{i\cdot}x \text{ for all } i \in [m]\right\}$$

with $\tilde{X} = \{(x,\bar{y}) \colon x \in X, D\bar{y} \geq b - Cx\}$, $\tilde{S}^i(x,\bar{y}) = \arg\min_{y'}\{B_{i\cdot}y' \colon Dy' \geq b - Cx, f^\top y' \leq f^\top \bar{y}\}$, *and* $y = (y^i)_{i=1}^m$.

Theorem

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$$\min_{(x,\bar{y})\in\tilde{X}} c^\top x + \min_{y} \left\{0 \colon y^i \in \tilde{S}^i(x,\bar{y}),\ B_{i\cdot}y^i \geq a_i - A_{i\cdot}x \text{ for all } i \in [m]\right\}$$

with $\tilde{X} = \{(x,\bar{y}) \colon x \in X, D\bar{y} \geq b - Cx\}$, $\tilde{S}^i(x,\bar{y}) = \arg\min_{y'}\{B_{i\cdot}y' \colon Dy' \geq b - Cx, f^\top y' \leq f^\top \bar{y}\}$, *and* $y = (y^i)_{i=1}^m$. *Let $\hat{\mathcal{S}}$ be the set of globally optimal solutions to the optimistic bilevel problem*

$$\min_{(x,\bar{y})\in\tilde{X}} c^\top x + \min_{y} \left\{0 \colon y \in \hat{S}(x,\bar{y}),\ B_{i\cdot}y^i \geq a_i - A_{i\cdot}x \text{ for all } i \in [m]\right\},$$

where $\hat{S}(x,\bar{y})$ denotes the set of optimal solutions to the aggregated lower-level problem

$$\min_{y} \quad \sum_{i=1}^{m} B_{i\cdot}y^i \quad \text{s.t.} \quad Dy^i \geq b - Cx,\ f^\top y^i \leq f^\top \bar{y} \quad \text{for all } i \in [m].$$

Theorem

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with $\tilde{X} = \{(x,\bar{y}) \colon x \in X, D\bar{y} \geq b - Cx\}$, $\tilde{S}^i(x,\bar{y}) = \arg\min_{y'}\{B_{i\cdot}y' \colon Dy' \geq b - Cx, f^\top y' \leq f^\top \bar{y}\}$, *and* $y = (y^i)_{i=1}^m$. *Let $\hat{\mathcal{S}}$ be the set of globally optimal solutions to the optimistic bilevel problem*

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where $\hat{S}(x,\bar{y})$ *denotes the set of optimal solutions to the aggregated lower-level problem*

$$\min_{y} \quad \sum_{i=1}^m B_{i\cdot}y^i \quad \text{s.t.} \quad Dy^i \geq b - Cx,\ f^\top y^i \leq f^\top \bar{y} \quad \text{for all } i \in [m].$$

Then, $\mathcal{S} = \mathrm{proj}_x(\tilde{\mathcal{S}}) = \mathrm{proj}_x(\hat{\mathcal{S}})$ *holds and all optimal objective function values coincide.*

Theorem (Simply apply the "old" optimistic result ...)

There is a poly-sized penalty parameter $\kappa > 0$ so that the optimistic bilevel problem with coupling constraints has the same set of globally optimal solutions as the optimistic bilevel problem

$$\min_{x \in X} \quad c^\top x + \min_{y,\varepsilon} \left\{ d^\top y + \kappa\varepsilon \colon (y, \varepsilon) \in S'(x) \right\}$$

without coupling constraints, where $S'(x)$ is the set of optimal solutions to the x-parameterized lower-level problem

$$\begin{aligned} \min_{y,\varepsilon} \quad & f^\top y \\ \text{s.t.} \quad & Ax + By + \varepsilon e \geq a, \\ & Cx + Dy \geq b, \\ & \varepsilon \geq 0, \end{aligned}$$

where e is the vector of all ones in appropriate dimension. Moreover, both bilevel problems have the same optimal objective function value.

Let's consider

$$\min_{(x,\bar{y})\in\tilde{X}} \quad F_{\mathrm{oa}}(x,\bar{y}) := c^\top x + d^\top \bar{y} + \min_{y,\varepsilon}\left\{0\colon \varepsilon = 0,\ (y,\varepsilon)\in\tilde{S}(x,\bar{y})\right\} \tag{AUX-UL}$$

with a single coupling constraint. Again, we use $\tilde{X} = \{(x,\bar{y})\colon x\in X, D\bar{y}\geq b - Cx\}$ and $\tilde{S}(x,\bar{y})$ denotes the set of optimal points to

$$\begin{aligned} \min_{y,\varepsilon} \quad & f^\top y \\ \text{s.t.} \quad & Cx + Dy \geq b, \quad f^\top \bar{y} - f^\top y = \varepsilon, \quad \varepsilon \geq 0. \end{aligned} \tag{AUX-LL}$$

Let's consider

$$\min_{(x,\bar{y})\in\tilde{X}} \quad F_{\mathrm{oa}}(x,\bar{y}) := c^\top x + d^\top \bar{y} + \min_{y,\varepsilon}\left\{0 \colon \varepsilon = 0,\ (y,\varepsilon) \in \tilde{S}(x,\bar{y})\right\} \tag{AUX-UL}$$

with a single coupling constraint. Again, we use $\tilde{X} = \{(x,\bar{y}) \colon x \in X, D\bar{y} \geq b - Cx\}$ and $\tilde{S}(x,\bar{y})$ denotes the set of optimal points to

$$\begin{aligned} \min_{y,\varepsilon} \quad & f^\top y \\ \text{s.t.} \quad & Cx + Dy \geq b, \quad f^\top \bar{y} - f^\top y = \varepsilon, \quad \varepsilon \geq 0. \end{aligned} \tag{AUX-LL}$$

Lemma

For every bilevel feasible point x of the optimistic bilevel problem without coupling constraints, the point $(x,\bar{y})$ with $\bar{y} \in \arg\min_y\{d^\top y \colon y \in S(x)\}$ is also bilevel feasible for the optimistic bilevel problem (AUX-UL) *with the same objective value. Moreover, for every globally optimal point $(x,\bar{y})$ to Problem* (AUX-UL), *x is bilevel feasible for the optimistic bilevel problem without coupling constraints with the same objective value.*

Lemma

There is a poly-sized parameter $\kappa > 0$ *so that Problem* (AUX-UL) *has the same set of globally optimal solutions as the optimistic bilevel problem*

$$\min_{(x,\bar{y}) \in \tilde{X}} \quad F_{\mathrm{o}\kappa}(x, \bar{y}) := c^\top x + d^\top \bar{y} + \min_{y,\varepsilon} \left\{ \kappa\varepsilon \colon (y, \varepsilon) \in \tilde{S}(x, \bar{y}) \right\}$$

without coupling constraints. Here, we again use $\tilde{X} = \{(x, \bar{y}) \colon x \in X, D\bar{y} \geq b - Cx\}$ *and* $\tilde{S}(x, \bar{y})$ *is the set of optimal solutions of* (AUX-LL).

Lemma

There is a poly-sized parameter $\kappa > 0$ so that Problem (AUX-UL) *has the same set of globally optimal solutions as the optimistic bilevel problem*

$$\min_{(x,\bar{y})\in\tilde{X}} \quad F_{\mathrm{o}\kappa}(x,\bar{y}) := c^\top x + d^\top \bar{y} + \min_{y,\varepsilon} \left\{\kappa\varepsilon \colon (y,\varepsilon) \in \tilde{S}(x,\bar{y})\right\}$$

without coupling constraints. Here, we again use $\tilde{X} = \{(x,\bar{y}) \colon x \in X, D\bar{y} \geq b - Cx\}$ and $\tilde{S}(x,\bar{y})$ is the set of optimal solutions of (AUX-LL)*.*

Theorem

For any κ, the optimistic bilevel problem without coupling constraints from the last lemma and its pessimistic version

$$\min_{(x,\bar{y})\in\tilde{X}} \quad F_{\mathrm{p}\kappa}(x,\bar{y}) := c^\top x + d^\top \bar{y} + \max_{y,\varepsilon} \left\{\kappa\varepsilon \colon (y,\varepsilon) \in \tilde{S}(x,\bar{y})\right\}$$

have the same set of feasible and globally optimal solutions.

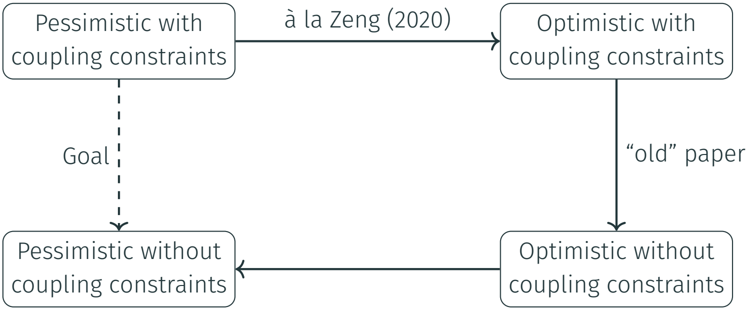
Corollary

There is a poly-sized parameter $\kappa > 0$ so that the optimistic bilevel problem without coupling constraints has the same set of globally optimal solutions as the pessimistic bilevel problem

$$\min_{(x,\bar{y}) \in \tilde{X}} \quad F_{\mathrm{p}\kappa}(x,\bar{y}) := c^\top x + d^\top \bar{y} + \max_{y,\varepsilon} \left\{ \kappa\varepsilon \colon (y,\varepsilon) \in \tilde{S}(x,\bar{y}) \right\}$$

without coupling constraints.

Bye bye, coupling constraints ...



Connections between robust and bilevel optimization

Question:

“What are the best and the worst aspects of this seminar?”

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“What are the best and the worst aspects of this seminar?”

Answer (by one of the participants):

“Best: Very active audience with a lot of questions.

Dagstuhl Seminar “Optimization at the Second Level” (2022): Survey Results

Question:

“What are the best and the worst aspects of this seminar?”

Answer (by one of the participants):

“Best: Very active audience with a lot of questions.

Worst: Found out that robust optimization is just a special case of bilevel optimization”

Later on Twitter

Jannis Kurtz
@JannisK13

I asked #chatgpt if #robustoptimization is a special case of #bileveloptimization .

Tweet übersetzen

JA Is robust optimization a special case of bilevel optimization?

No, robust optimization is not a special case of bilevel optimization. Robust optimization and bilevel optimization are two distinct types of optimization techniques that are used to solve different types of problems.

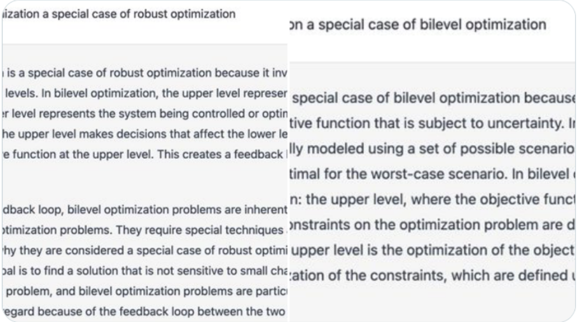
Robust optimization is a method of optimization that seeks to find solutions that are not overly sensitive to changes in the inputs or assumptions of the problem. This is achieved by introducing a certain degree of flexibility or "robustness" into the optimization process, allowing the solution to remain optimal even under a range of different conditions.

Bilevel optimization, on the other hand, is a type of optimization problem that involves two

Later on Twitter

Ryan Cory-Wright @RyanCoryWright · 9. Dez. 2022 ···

🤔



3 5

Jannis Kurtz @JannisK13 · 9. Dez. 2022 ···

Haha ok i think we solved P=NP now

2

Later on Twitter

R

why is bilevel optimization a special case of robust optimization

Bilevel optimization is a special case of robust optimization because it involves optimizing a problem with two levels. In bilevel optimization, the upper level represents the decision maker and the lower level represents the system being controlled or optimized. The decision maker at the upper level makes decisions that affect the lower level, which in turn affects the objective function at the upper level. This creates a feedback loop between the two levels.

Because of this feedback loop, bilevel optimization problems are inherently more complex than single-level optimization problems. They require special techniques and algorithms to solve, which is why they are considered a special case of robust optimization. In robust optimization, the goal is to find a solution that is not sensitive to small changes or uncertainties in the problem, and bilevel optimization problems are particularly challenging in this regard because of the feedback loop between the two levels.

Later on Twitter

R why is robust optimization a special case of bilevel optimization

Robust optimization is a special case of bilevel optimization because it involves the optimization of an objective function that is subject to uncertainty. In robust optimization, the uncertainty is typically modeled using a set of possible scenarios, and the goal is to find a solution that is optimal for the worst-case scenario. In bilevel optimization, there are two levels of optimization: the upper level, where the objective function is defined, and the lower level, where the constraints on the optimization problem are defined. In the case of robust optimization, the upper level is the optimization of the objective function, and the lower level is the optimization of the constraints, which are defined using the set of possible scenarios.

We wanted to find out!

Marc Goerigk

Jannis Kurtz

Johannes Thürauf

Optimistic Bilevel Optimization (à la Dempe 2002)

$$\begin{aligned} \min_{x,y} \quad & F(x,y) \\ \text{s.t.} \quad & G(x,y) \le 0 \\ & y \in S(x) \end{aligned}$$

$S(x)$: set of solutions to the x-parameterized problem

$$\begin{aligned} \min_{y} \quad & f(x,y) \\ \text{s.t.} \quad & g(x,y) \le 0 \end{aligned}$$

Pessimistic Bilevel Optimization (à la Wiesemann et al. 2013)

$$\begin{aligned} \min_{x} \quad & \max_{y \in S(x)} \; F(x, y) \\ \text{s.t.} \quad & G(x, y') \leq 0 \quad \forall y' \in S(x) \end{aligned}$$

$S(x)$: set of solutions to the x-parameterized problem

$$\begin{aligned} \min_{y} \quad & f(x, y) \\ \text{s.t.} \quad & g(x, y) \leq 0 \end{aligned}$$

Robust Optimization (Ben-Tal, El Ghaoui, Nemirovski; 2009)

$$\begin{aligned} \min_x \quad & H(x) \\ \text{s.t.} \quad & h_i(x, u_i) \leq 0 \quad \forall i \in I, \ \forall u_i \in U_i \\ & h_j(x) \leq 0 \quad \forall j \in J \end{aligned}$$

Important Case

Decision-dependent uncertainty: $U_i = U_i(x)$

Standard Assumptions

- For every robust feasible point x and every $i \in I$, the constraint functions $h_i(x, \cdot)$ are continuous
- For $i \in I$, the uncertainty set $U_i(x)$ is non-empty and compact

Main Research Question

If P is an instance of problem class $\mathcal{P}$

and if A is an algorithm for solving instances of problem class $\mathcal{Q}$,

can then A also be used to solve P?

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If P is an instance of problem class $\mathcal{P}$

and if A is an algorithm for solving instances of problem class $\mathcal{Q}$,

can then A also be used to solve P?

One can use an algorithm A

for solving optimistic bilevel optimization problems $\mathcal{Q}$

for solving a strictly robust optimization problem P.

Bilevel Methods can Solve Decision-Dependent Robust Problems

Theorem

Let the standard assumptions be satisfied. Let further (x^*, u^*) be a solution to the optimistic bilevel problem

$$\begin{aligned} \min_{x,u} \quad & H(x) \\ \text{s.t.} \quad & h_i(x, u_i) \leq 0 \quad \forall i \in I, \\ & h_j(x) \leq 0 \quad \forall j \in J, \\ & u \in S(x) \end{aligned}$$

where $S(x)$ is the set of solutions to the x-parameterized lower-level problem

$$\max_{u=(u_i)_{i\in I}} \quad \sum_{i \in I} h_i(x, u_i) \quad \text{s.t.} \quad u_i \in U_i(x) \; \forall i \in I.$$

Then, x^* is a solution to the strictly robust optimization problem with decision-dependent uncertainty sets $U_i(x)$, $i \in I$.

Some Remarks

- Both “standard” as well as decision-dependent uncertainty sets can be tackled
- Uncertain constraints with concave dependence on u and non-empty interior of the uncertainty sets lead to convex lower levels satisfying Slater’s CQ

Some Remarks

- Both “standard” as well as decision-dependent uncertainty sets can be tackled
- Uncertain constraints with concave dependence on u and non-empty interior of the uncertainty sets lead to convex lower levels satisfying Slater’s CQ
- The bilevel problem from the theorem is an optimistic one.
  Using a pessimistic one, the lower level can even have a constant objective function

$$
\begin{aligned}
\min_{x} \max_{u \in S(x)} \quad & H(x) \\
\text{s.t.} \quad & h_i(x, u_i) \leq 0 \quad \forall i \in I,\ \forall u = (u_i)_{i \in I} \in S(x) \\
& h_j(x) \leq 0 \quad \forall j \in J
\end{aligned}
$$

$S(x)$: set of solutions to the x-parameterized lower-level problem

$$
\min_{u=(u_i)_{i \in I}} \ 42 \quad \text{s.t.} \quad u_i \in U_i(x) \quad \forall i \in I
$$

And the other way around?

Theorem

Let (x^*, y^*) be a solution to the strictly robust problem

$$\begin{aligned} \min_{x,y} \quad & F(x,y) \\ \text{s.t.} \quad & f(x,y) \le f(x,\tilde{y}) \quad \forall \tilde{y} \in U(x), \\ & G(x,y) \le 0, \\ & g(x,y) \le 0, \end{aligned}$$

where the decision-dependent uncertainty set is given by

$$U(x) := \left\{ \tilde{y} \in \mathbb{R}^{n_y} : g(x,\tilde{y}) \le 0 \right\}.$$

Then, (x^*, y^*) is a solution to the optimistic bilevel problem.

First Main Result

Optimistic bilevel optimization

and

strictly robust optimization with decision-dependent uncertainty sets

are equivalent!

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PS: Both are equivalent to generalized semi-infinite optimization; see also Stein (2003)

What about pessimistic bilevel problems?

Theorem

A solution to the pessimistic bilevel problem can be computed by solving the following strictly robust problem with decision-dependent uncertainty set

$$\begin{aligned} \min_{x,y} \quad & F(x,y) \\ \text{s.t.} \quad & F(x,y) \geq F(x,y') \quad \forall y' \in U(x) \\ & G(x,y') \leq 0 \quad \forall y' \in U(x) \\ & f(x,y) \leq f(x,y') \quad \forall y' \in U(x) \\ & G(x,y) \leq 0 \\ & g(x,y) \leq 0 \end{aligned}$$

with uncertainty set

$$U(x) := \{\tilde{y} : f(x,\tilde{y}) \leq \chi(x),\ g(x,\tilde{y}) \leq 0\}.$$

Here, $\chi(x)$ is the optimal-value function of the x-parameterized lower-level problem.

Two-Stage Robust Optimization

- Uncertainty is still handled in a strict way
- Decisions are split
    - here-and-now
    - wait-and-see
- Ben-Tal, Goryashko, Guslitzer, Nemirovski (2004), Bertsimas, Den Hertog (2022)

$$\min_{x \in X} \max_{u \in U} \min_{y \in Y(x,u)} H(x, y)$$

with

$$Y(x, u) = \left\{ y \in \mathbb{R}^{n_y} : h(x, y, u) \leq 0 \right\}$$

Important difference:

1. Wait-and-see follower
2. Here-and-now follower

Robust bilevel problem with wait-and-see follower

$$\min_{x \in X} \max_{u \in U(x)} \min_{y} \; \{F(x, y) \colon y \in S(x, u)\}$$

$S(x, u)$: set of solutions to the (x, u)-parameterized problem

$$\min_{y} \quad f(x, y) \quad \text{s.t.} \quad g(x, y, u) \leq 0$$

Theorem

Let x^* be a solution to the optimistic robust bilevel problem with wait-and-see follower

$$\min_{x \in X} \max_{u \in U(x)} \min_{y} \{H(x, y) \colon y \in S(x, u)\}$$

where $X \subseteq \mathbb{R}^{n_x}$, $U(x) \subseteq \mathbb{R}^{n_u}$, and $S(x, u)$ is the set of solutions to the (x, u)-parameterized lower-level problem

$$\min_{y} \quad H(x, y) \quad \text{s.t.} \quad h(x, y, u) \leq 0.$$

Then, x^* is a solution to the two-stage robust problem with decision-dependent uncertainty set $U(x)$.

Robust Bilevel vs. Two-Stage-Robust Problems

Theorem

Let x^* be a solution to the optimistic robust bilevel problem with wait-and-see follower

$$\min_{x \in X} \max_{u \in U(x)} \min_{y} \{H(x,y) \colon y \in S(x,u)\}$$

where $X \subseteq \mathbb{R}^{n_x}$, $U(x) \subseteq \mathbb{R}^{n_u}$, and $S(x,u)$ is the set of solutions to the (x,u)-parameterized lower-level problem

$$\min_{y} \quad H(x,y) \quad \text{s.t.} \quad h(x,y,u) \leq 0.$$

Then, x^* is a solution to the two-stage robust problem with decision-dependent uncertainty set $U(x)$.

The other direction again requires using optimal-value functions.

Min-Max-Regret Optimization

$$
\begin{aligned}
\min_{x} \quad & \max_{u \in U} \left\{ H(x,u) - \min_{\{y:\ h(y,u) \leq 0\}} H(y,u) \right\} \\
\text{s.t.} \quad & h(x,u) \leq 0 \quad \forall u \in U
\end{aligned}
$$

Min-Max-Regret Optimization

$$\begin{aligned} \min_{x} \quad & \max_{u \in U} \left\{ H(x,u) - \min_{\{y:\ h(y,u) \leq 0\}} H(y,u) \right\} \\ \text{s.t.} \quad & h(x,u) \leq 0 \quad \forall u \in U \end{aligned}$$

Theorem

Let (x^*, y^*) be a solution to the pessimistic bilevel problem

$$\min_{x} \left\{ \max_{(y_1,y_2) \in S(x)} H(x,y_1) - H(y_2,y_1) \colon h(x,y_1') \leq 0 \ \forall y' = (y_1',y_2') \in S(x) \right\}$$

with $y = (y_1, y_2)$ and $S(x) = \arg\min_y \{42 : y_1 \in U, h(y_2,y_1) \leq 0\}$.

Then, x^* is a solution to the regret problem.

Min-Max-Regret Optimization

Min-max regret criterion is most commonly defined with uncertainty only in the objective

- Aissi et al. (2009), Kasperski and Zieliński (2016), Kouvelis and Yu (2013)

Theorem

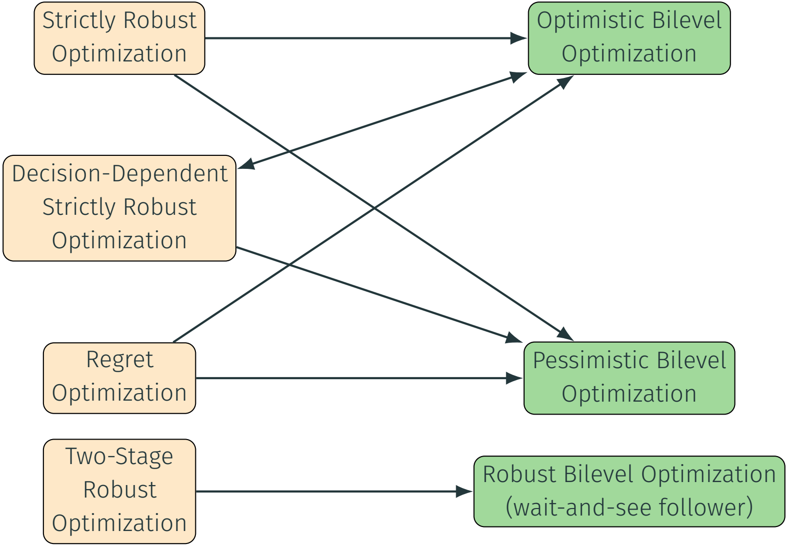
Let (x^*, y^*) *be a solution to the optimistic bilevel problem*

$$\min_{x, y \in S(x)} \{H(x, y_1) - H(y_2, y_1) \colon h(x) \leq 0\}$$

with $y = (y_1, y_2)$ *and* $S(x) = \arg\min_y \{H(y_2, y_1) - H(x, y_1) : y_1 \in U, h(y_2) \leq 0\}$.

Then, x^* *is a solution to the regret problem without uncertainty in the constraints.*

That's what we know now



An (in my opinion) important open problem

Smart People

$$
\begin{aligned}
\min_{x,y} \quad & F(x, y) \\
\text{s.t.} \quad & G(x, y) \leq 0 \\
& x \in \mathbb{R}^{n_x}, \quad y \in \mathbb{R}^{n_y} \\
& y \in S(x)
\end{aligned}
$$

$$\begin{aligned} \min_{x,y} \quad & F(x,y) \\ \text{s.t.} \quad & G(x,y) \leq 0 \\ & x \in \mathbb{R}^{n_x}, \quad y \in \mathbb{R}^{n_y} \\ & y \in S(x) \end{aligned}$$

$S(x)$: solution set of the convex lower-level problem

$$S(x) = \arg\min_{y} \left\{ f(x,y) \colon g(x,y) \leq 0,\ y \in \mathbb{R}^{n_y} \right\}$$

Once upon a time in multilevel gas market optimization ...

A “small” extension: black-box constraint in the lower level

$$S(x) = \arg\min_y \left\{ f(x,y) \colon g(x,y) \leq 0,\ b(y) \leq 0,\ y \in \mathbb{R}^{n_y} \right\}$$

A “small” extension: black-box constraint in the lower level

$$S(x) = \arg\min_{y} \{f(x,y) \colon g(x,y) \le 0,\ b(y) \le 0,\ y \in \mathbb{R}^{n_y}\}$$

Assumption

The black-box function b is convex and for all $(x,y) \in \{(x,y) \colon G(x,y) \le 0,\ g(x,y) \le 0\}$, ...

1. we can evaluate the function $b(y)$,
2. we can evaluate the gradient $\nabla b(y)$,
3. the gradient is bounded, i.e., $\|\nabla b(y)\| \le K$ for a fixed $K \in \mathbb{R}$.

A "First-Relax-Then-Reformulate" Approach

- Block-box constraint $b(y) \leq 0$ is convex
- Construct a sequence of linear outer approximations $(E^r, e^r)_{r \in \mathbb{N}}$ of the black-box constraint $b(y) \leq 0$ with the property

$$\{y \in \mathbb{R}^{n_y} : b(y) \leq 0\} \subseteq \{y \in \mathbb{R}^{n_y} : E^{r+1} y \leq e^{r+1}\} \subseteq \{y \in \mathbb{R}^{n_y} : E^r y \leq e^r\}$$

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- For a given upper-level solution $\bar{x} \in \Omega_{\mathrm{u}}$ and $r \in \mathbb{N}$, the adapted lower-level problem reads

$$\min_{y \in \mathbb{R}^{n_y}} \quad f(\bar{x}, y) \quad \text{s.t.} \quad g(\bar{x}, y) \leq 0, \; E^r y \leq e^r$$

- This is a relaxation of the original lower-level problem

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- Assumption: Slater's constraint qualification holds

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- For a given upper-level solution $\bar{x} \in \Omega_{\mathrm{u}}$ and $r \in \mathbb{N}$, the adapted lower-level problem reads

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- This is a relaxation of the original lower-level problem
- $\underline{\varphi}^r(x)$: optimal-value function
- Assumption: Slater's constraint qualification holds

Proposition

For every $r \in \mathbb{N}$ and every upper-level decision $x \in \Omega_{\mathrm{u}}$, it holds

$$\underline{\varphi}^r(x) \leq \underline{\varphi}^{r+1}(x) \leq \varphi(x).$$

“First-Relax-Then-Reformulate”

Choose $\delta_b > 0$, set $r = 0$, $s = 0$, $\chi = \infty$, $E^0 = [0 \dots 0] \in \mathbb{R}^{1 \times n_y}$, $e^0 = 0 \in \mathbb{R}$.
while $\chi > \delta_b$ or $s > 0$ **do**
  Construct E^{r+1} and e^{r+1}.
  if the modified variant of the single-level reformulation is feasible **then**
    Solve this problem to obtain (x^{r+1}, y^{r+1}) and set $s = 0$.
  else if the feasibility problem is feasible **then**
    Solve this problem to obtain (x^{r+1}, y^{r+1}, s).
  else
    Return “The original problem is infeasible.”
  end if
  Set $r \leftarrow r + 1$ and $\chi = b(y^r)$.
end while
Return $(\bar{x}, \bar{y}) = (x^r, y^r)$.

“First-Relax-Then-Reformulate”

Choose $\delta_b > 0$, set $r = 0$, $s = 0$, $\chi = \infty$, $E^0 = [0 \dots 0] \in \mathbb{R}^{1 \times n_y}$, $e^0 = 0 \in \mathbb{R}$.
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  else
    Return “The original problem is infeasible.”
  end if
  Set $r \leftarrow r + 1$ and $\chi = b(y^r)$.
end while
Return $(\bar{x}, \bar{y}) = (x^r, y^r)$.

Theorem: If the algorithm terminates, then $(\bar{x}, \bar{y})$ is $(0, 0, \delta_b, 0)$-feasible for original bilevel problem.

All the details ...

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On convex lower-level black-box constraints in bilevel optimization with an application to gas market models with chance constraints

Holger Heitsch, René Henrion, Thomas Kleinert & Martin Schmidt

Journal of Global Optimization **84**, 651–685 (2022) | Cite this article

Abstract

Bilevel optimization is an increasingly important tool to model hierarchical decision making. However, the ability of modeling such settings makes bilevel problems hard to solve in theory and practice. In this paper, we add on the general difficulty of this class of problems by further incorporating convex black-box constraints in the lower level. For this setup, we develop a cutting-plane algorithm that computes approximate bilevel-feasible points. We apply this method to a bilevel model of the European gas market in which we use a joint chance constraint to model uncertain loads. Since the chance constraint is not available in closed form, this fits into the black-box setting studied before. For the applied model, we use further problem-specific insights to derive bounds on the objective value of the bilevel problem. By doing so, we are able to show that we solve the application problem to approximate global optimality. In our numerical case study we are thus able to evaluate the welfare sensitivity in dependence of the achieved safety level of uncertain load coverage.

Nonconvexities in the Lower Level

Upper-level problem

$$\begin{aligned} \text{“}\min_x\text{”} \quad & F(x,y) \\ \text{s.t.} \quad & G(x,y) \geq 0, \quad y \in S(x) \end{aligned}$$

Lower-level problem

$$\begin{aligned} \min_y \quad & f(x,y) \\ \text{s.t.} \quad & g(x,y) \geq 0 \end{aligned}$$

Who can solve this problem?

Upper level

$$\begin{aligned} \max_{x \in \mathbb{R}^2} \quad & F(x, y) = x_1 - 2y_{n+1} + y_{n+2} \\ \text{s.t.} \quad & (x_1, x_2) \in [\underline{x}_1, \bar{x}_1] \times [\underline{x}_2, \bar{x}_2] \\ & y \in S(x) \end{aligned}$$

- $\underline{x}$, $\bar{x} \in \mathbb{R}^2$ with $1 \leq \underline{x}_i < \bar{x}_i$, $i \in \{1, 2\}$
- Upper level is an LP
  with simple bound constraints
- Upper level has no coupling constraints

Who can solve this problem?

Upper level

$$\begin{aligned} \max_{x \in \mathbb{R}^2} \quad & F(x,y) = x_1 - 2y_{n+1} + y_{n+2} \\ \text{s.t.} \quad & (x_1, x_2) \in [\underline{x}_1, \bar{x}_1] \times [\underline{x}_2, \bar{x}_2] \\ & y \in S(x) \end{aligned}$$

Lower level

$$\begin{aligned} \max_{y \in \mathbb{R}^{n+2}} \quad & f(x,y) = y_1 - y_n \left(x_1 + x_2 - y_{n+1} - y_{n+2}\right) \\ \text{s.t.} \quad & y_1 + y_n = \frac{1}{2} \\ & y_i^2 \leq y_{i+1}, \quad i \in \{1, \dots, n-1\} \\ & y_i \geq 0, \quad i \in \{1, \dots, n\} \\ & y_{n+1} \in [0, x_1] \\ & y_{n+2} \in [-x_2, x_2] \end{aligned}$$

- $\underline{x}, \bar{x} \in \mathbb{R}^2$ with $1 \leq \underline{x}_i < \bar{x}_i$, $i \in \{1, 2\}$
- Upper level is an LP with simple bound constraints
- Upper level has no coupling constraints
- Feasible set of lower level is non-empty and compact for all feasible leader decisions
- Slater's CQ is also satisfied for all feasible leader decisions
- All constraints are linear except for some convex-quadratic inequality constraints
- The coefficients/right-hand sides are either 0, 1, or 1/2
- Bilinear objective function

Exact Feasibility

$$\max_{y \in \mathbb{R}^{n+2}} \quad f(x,y) = y_1 - y_n\,(x_1 + x_2 - y_{n+1} - y_{n+2})$$

$$\text{s.t.} \quad y_1 + y_n = \frac{1}{2}$$

$$y_i^2 \leq y_{i+1}, \quad i \in \{1, \ldots, n-1\}$$

$$y_i \geq 0, \quad i \in \{1, \ldots, n\}$$

$$y_{n+1} \in [0, x_1]$$

$$y_{n+2} \in [-x_2, x_2]$$

Result #1

For every feasible leader's decision $(x_1, x_2) \in [\underline{x}_1, \bar{x}_1] \times [\underline{x}_2, \bar{x}_2]$, a feasible follower's decision y satisfies $y_n > 0$.

Exact Feasibility

$$
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Result #2

For every feasible leader's decision $(x_1, x_2) \in [\underline{x}_1, \bar{x}_1] \times [\underline{x}_2, \bar{x}_2]$, the set of optimal solutions of the lower-level problem is a singleton.

Exact Feasibility

$$
\begin{aligned}
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\text{s.t.} \quad & y_1 + y_n = \frac{1}{2} \\
& y_i^2 \leq y_{i+1}, \quad i \in \{1, \dots, n-1\} \\
& y_i \geq 0, \quad i \in \{1, \dots, n\} \\
& y_{n+1} \in [0, x_1] \\
& y_{n+2} \in [-x_2, x_2]
\end{aligned}
$$

Result #1

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Result #2

For every feasible leader's decision $(x_1, x_2) \in [\underline{x}_1, \bar{x}_1] \times [\underline{x}_2, \bar{x}_2]$, the set of optimal solutions of the lower-level problem is a singleton.

Result #3

The bilevel problem has a unique solution given by $x^* = (\underline{x}_1, \bar{x}_2)$ with an optimal objective function value of $F^* = \underline{x}_1 + \bar{x}_2$.

Definition

Let $0 < \varepsilon \in \mathbb{R}$, $f : \mathbb{R}^n \to \mathbb{R}$, and $g : \mathbb{R}^n \to \mathbb{R}^m$ be given. A point $x \in \mathbb{R}^n$ is called ε-feasible for the problem $\max_{x \in \mathbb{R}^n} \{f(x) \colon g(x) \leq 0\}$ if $g_i(x) \leq 0$ holds for all $i \in \{1, \ldots, m\} \setminus N$ and if $\max_{i \in N} g_i(x) \leq \varepsilon$ holds, where $N \subseteq \{1, \ldots, m\}$ denotes the index set of all nonlinear constraints.

Definition

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Result #4

Unless $\varepsilon < 2^{-2^{n-1}}$, there is an ε-feasible follower's decision y with $y_n = 0$ for every feasible leader's decision $(x_1, x_2) \in [\underline{x}_1, \bar{x}_1] \times [\underline{x}_2, \bar{x}_2]$.

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Result #5

Unless $\varepsilon < 2^{-2^{n-1}}$, the set of ε-feasible follower's solutions is not a singleton for every feasible leader's decision $(x_1, x_2) \in [\underline{x}_1, \bar{x}_1] \times [\underline{x}_2, \bar{x}_2]$.

Result #3 (revisited)

The bilevel problem has a unique solution given by $x^* = (\underline{x}_1, \bar{x}_2)$ with an optimal objective function value of $F^* = \underline{x}_1 + \bar{x}_2$.

ε-feasibility

Result #3 (revisited)

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Result #6

Let $\varepsilon \geq 2^{-2^{n-1}}$ and suppose that we allow for ε-feasible follower's solutions.

Then, the optimistic optimal solution of the bilevel problem is given by $x_{\mathrm{o}}^* = (\bar{x}_1, \bar{x}_2)$ with an optimal objective function value of $F_{\mathrm{o}}^* = \bar{x}_1 + \bar{x}_2$.

The pessimistic optimal solution is given by $x_{\mathrm{p}}^* = (\underline{x}_1, \underline{x}_2)$ with an optimal objective function value of $F_{\mathrm{p}}^* = -\underline{x}_1 - \underline{x}_2$.

ε-feasibility

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- By the way: $n \geq \log_2(\log_2(1/\varepsilon^2))$
- For $\varepsilon = 10^{-8}$, the problem gets unsolvable for $n = 6$

Well ... and now?

Is this an impossibility result
for computationally solving bilevel problems
with continuous and nonconvex lower-level problems?

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Lefebvre, Schmidt, Svensson, Thürauf:

Calmness of the Solution-Set Mapping for Linear Bilevel and Pricing Problems

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\begin{aligned}
\min_{x,y} \quad & c^\top x + d^\top y \\
\text{s.t.} \quad & Ax + By \geq a - \varepsilon^u \\
& y \in S_{\varepsilon^l}(x)
\end{aligned}
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where $S_{\varepsilon^l}(x)$ is the set of optimal solutions to the ε^l-perturbed lower-level problem given by

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Calmness at 0: $\exists$ a constant $\kappa > 0$ so that for any $(x_\varepsilon^*, y_\varepsilon^*) \in S(\varepsilon)$ with ε sufficiently small, there exists $(x^*, y^*) \in S(0)$ such that

$$
\|(x_\varepsilon^*, y_\varepsilon^*) - (x^*, y^*)\| \leq \kappa \|\varepsilon\|
$$

Calmness in Linear Bilevel Optimization

Problem Class	Solution-Set Mapping $\varepsilon \mapsto S(\varepsilon)$: Outer Semicont. at 0	Solution-Set Mapping $\varepsilon \mapsto S(\varepsilon)$: Calmness at 0
LB without coupling constraints Perturbations: $(\varepsilon^u, \varepsilon^l) \in \mathbb{R}^{m+\ell}$	✓	✓
LB with coupling constraints Perturbations: $\varepsilon^u = 0, \varepsilon^l \leq 0$	✗	✗
LB with coupling constraints Perturbations: $\varepsilon^u = 0, \varepsilon^l \geq 0$	✗	✗
LB with coupling constraints Perturbations: $\varepsilon^u \leq 0, \varepsilon^l = 0$	✗	✗
LB with coupling constraints Perturbations: $\varepsilon^u \geq 0, \varepsilon^l = 0$	✓	✓

Calmness for Bilevel Pricing

	Solution-Set Mapping $\varepsilon \mapsto S(\varepsilon)$	
Perturbations	Outer Semicont. at 0	Calmness at 0
$\varepsilon^u \leq 0, \varepsilon^l = 0$	✗	✗
$\varepsilon^u \geq 0, \varepsilon^l \in \mathbb{R}^\ell$	✓	✓

Calmness for Bilevel Pricing

	Solution-Set Mapping $\varepsilon \mapsto S(\varepsilon)$	
Perturbations	Outer Semicont. at 0	Calmness at 0
$\varepsilon^u \leq 0, \varepsilon^l = 0$	✗	✗
$\varepsilon^u \geq 0, \varepsilon^l \in \mathbb{R}^\ell$	✓	✓

Work in progress: nonlinear cases

The End

There is a lot more to discover and to study!

- Bilevel optimization with discrete variables
- Bilevel optimization with nonlinear lower-level problems
- Stochastic bilevel optimization
- Robust bilevel optimization
- Bounded rationality
- etc. etc. etc.

Background Material

Linear and Mixed-Integer Bilevel Optimization: Theory and Algorithms

Jointly with Ivana Ljubic and Yasmine Beck

Free pre-publication version

Scan me!

BOBILib: Bilevel Optimization (Benchmark) Instance Library

- More than 2600 instances of mixed-integer linear bilevel optimization problems
- Well-curated set of test instances
- Freely available for the research community
- Testing of new methods + comparison with other ones
- Different types of instances
  - Interdiction
  - Mixed-integer
  - Pure integer
- Benchmark sets for all of them
- Extensive numerical results
- New data + solution format
- All best known solutions available

`https://bobilib.org`