From Minimum Cuts to Submodular Systems (and back)

Standing on the Shoulders of Giants

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In the Beginning…

…worked under the mentorship of **Jean-Claude Picard** (1938-1999) on **minimum (*s*,*t*)-cuts** in networks – learning from some earlier giants:

- Maxflow Mincut Theorem [P. Elias, A. Feinstein, & **C.E. Shannon**, 1956; **L.R. Ford, Jr.** & **D.R. Fulkerson**, 1956]
- Minimum (*s*,*t*)-cuts as instances of pseudo-Boolean programming [**P.L.Hammer** (Ivanescu), 1965]
- Binary maximization of quadratic polynomials with nonnegative quadratic coefficients as minimum (*s*,*t*)-cuts problems [J.-C. Picard & H.D. Ratliff, 1975]
- Maximum weight closure of a graph (poset ideal or filter) as a minimum (*s*,*t*)-cut problem, and application to open pit mining [J.-C. Picard, 1976]

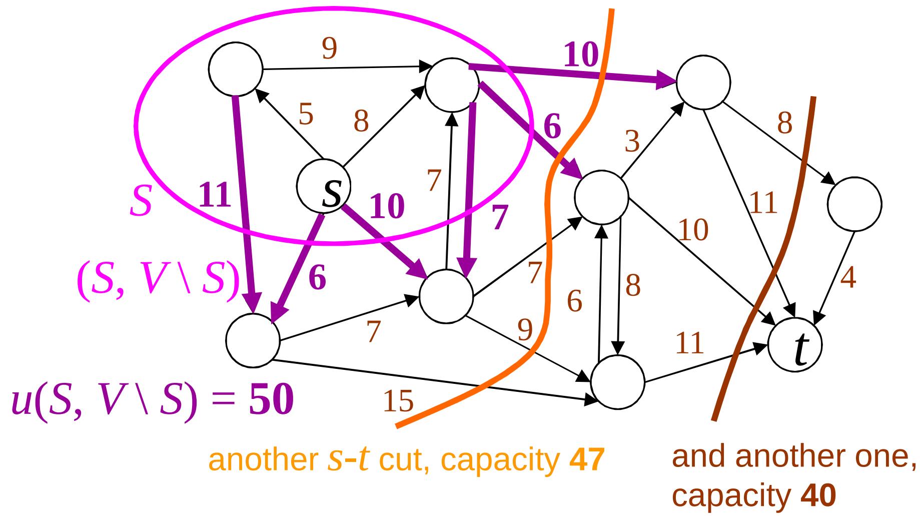
Minimum (s,t) cuts

In a directed graph $G = (V, A)$ with

- a source s and a sink t in V $(s \neq t)$ and
- arc capacities $u_{ij} \geq 0$ for all $ij \in A$

an **(s,t)-cut** $(S, V \setminus S)$ is the set of all arcs $ij \in A$ with $i \in S$ and $j \notin S$, with source set $S \subset V$ containing s and not t

- its capacity is $u(S, V \setminus S) = \sum_{ij \in (S, V \setminus S)} u_{ij}$



[P.L.Hammer (Ivanescu), 1965]: let binary variable $x_i = 1$ if $i \in S$, and 0 otherwise

$$u(S, V \setminus S) = \sum_{ij \in A} u_{ij}\, x_i\, (1 - x_j)$$

[J.-C. Picard & H.D. Ratliff, 1975]: conversely, every quadratic polynomial

$$f(x) = ½\, x^T Q\, x + b^T x$$

in n binary variables with all non-diagonal quadratic coefficients $q_{ij} \geq 0$ $(i \neq j)$ may be maximized by solving a minimum (s,t)-cut problem

Minimum (*s*,*t*)-cuts

Joint work with **Jean-Claude Picard** on minimum (*s*,*t*)-cuts

- *On the structure of all minimum cuts in a network and applications* (1980): the lattice structure of minimum cuts
- *A network flow solution to some nonlinear 0-1 programming problems, with applications to graph theory* (1982): hyperbolic optimization problems and nested parametric minimum cuts
- *Selected applications of minimum cuts in networks* (1982), from the binary quadratic programming formulation and parametric properties
- *Ranking the cuts and cut-sets of a network* (with H. Hamacher, 1984)

Minimum (*s*,*t*)-cuts as instances of algebraic lattices

- The intersection $S \cap S'$ and the union $S \cup S'$ of two minimum (*s*,*t*)-cuts *S* and *S*´ are also minimum (*s*,*t*)-cuts [Ø. Ore, 1962]
- An (algebraic) **lattice** is a poset (partially ordered set) in which every two elements *x* and *y* have a greatest common lower bound, their meet $x \wedge y$, and a smallest common upper bound, their join $x \vee y$
- Structure of sublattices of **product spaces** (products of chains, such as $\mathbf{R}^n$ or $\mathbf{Z}^n$) [D.M. Topkis, 1976; A.M. (Pete) Veinott, Jr., 1989]

Joint work with Fabio Tardella on the structure of sublattices:

- *Bimonotone linear inequalities and sublattices of* $\mathbf{R}^n$ (2006): characterize closed convex sublattices of $\mathbf{R}^n$
- *Sublattices of product spaces: Hulls, representation and counting* (2008): representations with proper boundary epigraphs allow counting sublattices of finite products of finite chains, and yield a good characterization and a polytime algorithm for sublattice hull membership
- *Carathéodory, Helly and Radon numbers for sublattice convexities* (t.a.): exact or approximate values of convexity invariants for several convexities defined by sublattices of $\mathbf{B}^n$, $\mathbf{R}^n$ and $\mathbf{Z}^n$

(s,t)-cut functions are submodular

The cut (capacity) function $f : 2^V \to \mathbf{R}$, where $f(S) = u(S, V \setminus S)$ is the capacity of the (s,t)-cut defined by S, satisfies the **submodular inequality**

$$f(S \cap S') + f(S \cup S') \leq f(S) + f(S') \quad \text{for all } S, S' \subset V$$

Pioneering work of Jack Edmonds (1970) on submodular set functions, greedy algorithms and polymatroids (also L. Lovász, S. Fujishige, etc.)

- *A general class of greedily solvable linear programs* (with F. Spieksma & F. Tardella, 1998): duality relationship with transportation problems satisfying a Monge condition

Applications to **sequencing and scheduling**:

- *Structure of a simple scheduling polyhedron* (1993): Smith's rule (WSPT, c-μ rule) is an instance of the polymatroid greedy algorithm in disguise (see also [L. Wolsey, 1985])
- *Single machine scheduling with release dates* (with M. Goemans, A. Schulz, M.Skutella & Y. Wang, 2002)
- *Approximation algorithms for shop scheduling problems with minsum objective* (with M.Sviridenko, 2002)
- *On the asymptotic optimality of a simple on-line algorithm for the stochastic single machine weighted completion time problem and its extensions* (with C. Chou, H. Liu & D. Simchi-Levi, 2006)

Parametric minimum (*s*,*t*)-cuts

Let the arc capacities $u_{ij}(\lambda)$ be functions of a parameter λ

Two key properties:

- Structural property: if the capacities of the source arcs (*s*,*v*) are increasing functions of λ and those of the sink arcs (*v*,*t*) decreasing functions of λ then minimum (*s*,*t*)-cuts $S(\lambda)$ are increasing (nested) [M.J. Eisner & D.G. Severance, 1976; H.S. Stone, 1978; Picard & Q. 1982]

an instance of parametric submodular optimization [D.M. Topkis, 1978]

- Algorithmic property: full parametric analysis in about the same time as a single minimum (*s*,*t*)-cut computation [G. Gallo, M.D. Grigoriadis & R.E.Tarjan, 1989, known as "*GGT*"]

Further extensions of structural and algorithmic properties:

- *Monotone parametric min cut revisited: Structures and algorithms* (with F.Granot, S.T. McCormick & F. Tardella, 2012)

Further applications of minimum (*s*,*t*)-cuts

- *A study of the Bienstock-Zuckerberg algorithm: Applications in mining and resource constrained project scheduling* (with G. Muñoz, D. Espinoza, M. Goycoolea, E. Moreno & O. Rivera, submitted)

The Maxflow-Mincut Thm is a special case of the Kantorovich Duality of (infinite dimensional) optimal transport problems:

- *Optimal pits and optimal transportation* (with I. Ekeland, 2015): a continuous space optimum closure model
- *Combinatorial bootstrap inference in partially identified incomplete structural models* (with M. Henry & R. Meango, 2015): an application to econometrics (extending work of I. Ekeland, A. Galichon & M. Henry)

Global minimum cuts and symmetric submodular functions

- There are simpler and faster algorithms for finding global minimum cuts in undirected networks than solving $|V|$-1 minimum (s,t)-cut problems [H.Nagamochi &T. Ibaraki, 1992; D.R. Karger, 1993; D.R. Karger & C.Stein, 1993]

Main thesis [Q 1999]: "*most properties of global cuts are properties of* **symmetric** submodular functions"

  - i.e., submodular set functions such that $f(S) = f(V \setminus S)$ for all $S \subseteq V$

- *Minimizing symmetric submodular functions* (1998): a genuinely combinatorial $O(|V|^3)$ algorithm, extending the MA ordering approach of Nagamochi & Ibaraki
  - Renewed interest in seeking combinatorial (non-ellipsoid) algorithms for minimizing (general) submodular set functions [A. Schrijver [*EURO Gold Medal 2015*], 2000; S. Iwata, L. Fleischer & S. Fujishige, 2001; etc.]
  - Applied to statistical physics [J.-Ch. Anglès d'Auriac, F. Iglói, M. Preissmann & A. Sebö, 2002] and clustering [M. Narasimhan, N. Jojic & J. Bilmes, 2005]

Parametric global minimum cuts

- Parametric global minimum cuts lack the Structural and Algorithmic properties of parametric minimum (s,t)-cuts

However, whereas virtually every parametric combinatorial optimization problem has a super-polynomial number of (Pareto) efficient solutions (including minimum (s,t) cuts [P. Carstensen, 1983]), parametric global minimum cuts have a polynomial number of efficient solutions, and polytime algorithms for enumerating them; e.g., for a single parameter λ:

- About $O(|V|^{19})$ efficient solutions [K. Mulmuley, 1999]
- *Strongly polynomial bounds for multiobjective and parametric global minimum cuts in graphs and hypergraphs* (with H. Aissi, R. Mahjoub, S.T. McCormick, 2015): $O(|V|^7)$ efficient solutions
- Improved algorithms and insights [Karger, 2016]

Thank you for your interest and attention

…and my apologies to the many contributors I did not cite