

Bi-Objective Approach to Parametric Max-Flow Interdiction

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We present a new algorithm for solving a maximum flow network interdiction problem parametrized in an interdictor's available resources. Interdiction problems are leader-follower games in which the leader selects interdiction strategies to minimize the maximum objective value attainable by the follower in its optimization task. Such problems arise in applications related to infrastructure reliability and vulnerability analysis, natural disaster response, and others. There exists some research considering interdictions against general optimization tasks like linear programs or integer programs, see [6] and [1]. However, research has primarily focused on network interdiction problems, as noted in the survey by Smith and Song [4]. Here, the follower's task is to solve a network optimization problem and the leader uses limited resources to manipulate the network, for example, by deleting edges or by adjusting attributes like lengths, capacities, or costs. In [5], Wood considers the interdiction of a maximum flow in a capacitated network. Motivated by real world decision making and the necessity to consider cost-benefit analysis, Royset and Wood [3] extend the problem to bi-objective maximum flow network interdiction. Thus, they derive interdiction strategies that do not only minimize the resulting flow, but, at the same time, minimize the amount of resource to do so. The goal is to find all Pareto-optimal solutions, that is, solutions for which there is no other solution, that is better in both objectives at the same time.

In this work, we consider a related, but distinct problem from [3]: While Royset and Wood focus on binary interdiction decisions resulting in a finite Pareto-front, our problem allows for continuous decisions which enables the leader to linearly decrease edge capacities. Phrasing it in terms of bi-objective optimization, we deal with a continuous, thus infinite, set of Pareto-optimal solutions for which we present an exact algorithm to compute them. We characterize the so-called objective value function of our parametric problem, mapping the parametrized interdictor's resources to the resulting objective value, in terms of the Pareto-optimal solutions of a related bi-objective problem.

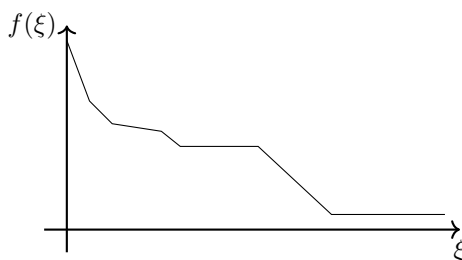


Figure 1: Illustration of an objective value function.

Let N be a set of nodes, and $A \subset N \times N$ be a set of arcs. The resulting directed graph is denoted by $G = (N, A)$. We identify the special nodes $s \neq t$, where s is the source node and t is the sink node. Each arc $k \in A$ has a capacity $u_k \in \mathbb{Z}_{>0}$ limiting the flow over k , an interdiction cost $\varphi_k \in \mathbb{Z}_{>0}$ and an interdiction bound $r_k \in \mathbb{Z}_{\geq 0}$. The interdiction cost φ_k is the amount of resource necessary to decrease the capacity of the arc k by one. The interdiction bound $r_k \leq u_k$ limits the resource, that can be used to interdict an arc. By $\Theta = \{z \in \mathbb{R}^{|A|} \mid 0 \leq z_k \leq r_k \forall k \in A\}$ we denote the set of feasible interdiction strategies and the cost of a strategy z is $c(z) := \sum_{k \in A} \varphi_k z_k$. The resulting network with respect to a chosen interdiction strategy z is denoted by $G(z) = (N, A)$ with adjusted arc capacities $u'_k = u_k - z_k$ and

a maximal flow, denoted by $g(z)$. We formulate the parametric max-flow interdiction problem as

$$\begin{aligned} \text{[PMFI]} \quad f(\xi) &:= \min_{z \in \Theta} g(z) \\ &\text{s.t.} \quad c(z) \leq \xi \quad \forall \xi \in \mathbb{R}_{\geq 0}, \end{aligned}$$

where f refers to the objective value function.

A cut $C \subset A$ is a set of arcs, such that deleting all arcs in C disconnects s from t . The strong duality between maximal flows and minimal cuts allows us to reformulate PMFI to a problem in the leader's interdiction variable z and a follower's cut-variable C . Then, the value of C , $v(C, z) = \sum_{k \in C} u'_k$ does depend on z , but the feasibility does not. Thus, in terms of feasibility, the leader and the follower can decide independently of one another. We use this observation for the analysis of the objective value function f . It is a function in $\mathbb{R}_{\geq 0}$, where the domain corresponds to the cost of the leader's strategy $c(z)$ and the function value corresponds to the maximum flow in the interdicted network. We show that it is monotonously decreasing and piecewise linear with the line segment slopes depending on the φ -values (see Figure 1 for an illustration). Moreover, we show that in general f is neither concave nor convex. Using the series-parallel graph in Figure 2 with $2+n$ nodes and $2n$ arcs, we show that even for unit costs $\varphi_k = 1$, the number of breakpoints of the value function can be exponential in the size of the problem instance.

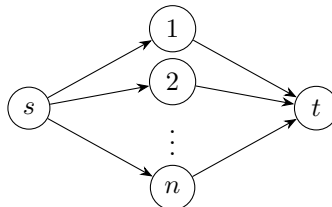


Figure 2: A graph yielding and instance with exponentially many breakpoints.

Generally, so-called dichotomic search methods from multi-objective optimization are only capable of computing the convex hull of the Pareto-front, cf. [2]. However, we adapt a variant of it to be able to compute all the line segments of the objective value function f . Methodologically, we compute upper and lower bounds for the complete objective value function, which we iteratively tighten in each step of the algorithm. Using weighted sums, we tighten the lower bounds, eventually proving optimality of linear segments of the upper bound. Then, we compute feasible solutions for each interdiction cost, generating valid lower bounds on the objective value function. Further research includes extending the algorithm's framework to approximate the objective value function.

References

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