Automated Benders-like Cut Generation and its Application to the Bilevel Network Design Problem

Vladimir Stadnichuk[1] and Arie M.C.A. Koster[2]

[1]Research Group on Algorithmic Algebra and Discrete Mathematics, University of Kassel, Kassel, Germany, ✉ vladimir.stadnichuk@uni-kassel.de
[2]Discrete Optimization Teaching and Research Unit, RWTH Aachen University, Aachen, Germany, ✉ koster@math2.rwth-aachen.de

We study bilevel network design problems in which a first-level network operator makes binary decisions on which arcs to include in the network, while a set of independent users at the second level solve routing problems on the resulting network. A standard solution approach is to apply Benders-like decomposition [1, 2, 3] to reformulate the bilevel model as a single-level mixed-integer problem that can be handled by off-the-shelf MIP solvers. This reformulation is based on approximating the second-level value function by linear inequalities, so-called Benders-like cuts, which are generated iteratively. The approach starts with a High Point Relaxation, i.e., an approximation in which the second level is only required to choose a feasible, but not necessarily optimal, solution. Whenever the solution chosen by the second level is not bilevel-feasible, i.e., not optimal with respect to its objective, we add a Benders-like cut that restores bilevel feasibility for the current first-level decision. The process is iterated till no violated constraint is found.

The main challenge in using Benders-like decomposition is to derive high-quality coefficients for the Benders-like cuts that accurately capture the impact of first-level decisions on the second-level value function. To the best of our knowledge, no general framework exists for deriving such coefficients in bilevel network design problems. As a consequence, one often resorts either to very large constants, which lead to formulations close to full enumeration, or to highly problem-specific structures that are difficult to generalize.

Against this background, our main contribution is an automated approach for generating Benders-like cut coefficients for bilevel network design problems. Our idea build upon our recent work [6], and naturally extends the recent approach of Zou et al. [7] for multistage optimization to the bilevel setting. In their work, so-called Lagrangian cuts approximate value functions, with cut coefficients obtained from an associated Lagrangian dual. We transfer this idea to Benders-like cuts by proposing a novel Lagrangian dual whose optimal solutions yield valid cut coefficients. Due to the underlying bilevel structure, the resulting Lagrangian subproblem is itself a bilevel problem, which is consistent with the well-known fact that finding optimal Benders-like cut coefficients is essentially as hard as solving the original bilevel problem [4]. However, for bilevel network design problems with multiple independent users, our Lagrangian dual decomposes by user, reducing the original problem to a sequence of smaller bilevel problems needed to generate the Benders-like cut coefficients.

We implement our approach in the publicly available open-access Julia package JuBiC (`https://github.com/vstadnichuk/JuBiC`) and conduct computational experiments on bilevel network design instances with different structures. To further improve computational performance, we use column generation to solve the Lagrangian dual [5], which allows us to warm-start the solution process between iterations. Preliminary results indicate that the approach is limited in settings where the problem structure forces near-enumeration of all second-level solutions in the Lagrangian dual, but shows promising performance in scenarios where the first-level network operator faces strong competition that offers high-quality alternative services to the users.

References

[1] E. Israeli. *System interdiction and defense.* PhD thesis, Naval Postgraduate School, 1999.

[2] Eitan Israeli and R. Kevin Wood. Shortest-path network interdiction. *Networks*, 40(2):97–111, 2002.

[3] Thomas Kleinert, Martine Labbé, Ivana Ljubić, and Martin Schmidt. A Survey on Mixed-Integer Programming Techniques in Bilevel Optimization. *EURO Journal on Computational Optimization*, 9:100007, 2021.

[4] Thomas Kleinert, Martine Labbé, Fränk Plein, and Martin Schmidt. Technical note—there's no free lunch: On the hardness of choosing a correct big-M in bilevel optimization. *Operations Research*, 68(6):1716–1721, 2020.

[5] Marco E. Lübbecke. Column generation, 2011.

[6] Vladimir Stadnichuk and Arie M. C. A. Koster. Solving multi-follower mixed-integer bilevel problems with binary linking variables. Optimization Online, 2024.

[7] Jikai Zou, Shabbir Ahmed, and Xu Andy Sun. Stochastic dual dynamic integer programming. *Mathematical Programming*, 175(1–2):461–502, 2019.