

On mixed-integer programming approaches for the α -neighbor p -center problem

Elisabeth Gaar¹ and Markus Sinnl²

¹Institute of Mathematics, University of Augsburg, Augsburg, Germany, ✉ elisabeth.gaar@uni-a.de

²Institute of Business Analytics and Technology Transformation/JKU Business School, Johannes Kepler University, Linz, Austria, ✉ markus.sinnl@jku.at

Abstract

We investigate the α -neighbor p -center problem, a recently emerged generalization of the classical p -center problem and present exact solution algorithms based on mixed-integer programming. Three formulations, along with valid inequalities, lifting procedures, and variable-fixing techniques, are introduced, and a polyhedral study is done. Based on these insights, we develop branch-and-cut algorithms whose effectiveness is demonstrated on standard benchmark instances from literature.

Keywords: p -center problem; mixed-integer linear programming; facility location

The α -neighbor p -center problem (α - p CP), first proposed in Krumke [3], is a relatively recent extension of the classical p -center problem (p CP) and has attracted growing interest in the literature. In the α - p CP, a set of points is given and the task is to place p facilities. The objective is to choose facility locations such that the maximum distance between any point and its α -closest facility is minimized. The α - p CP can be interpreted as a robust generalization of the p CP, as customers are not restricted to being served by their nearest facility but instead have access to α nearby facilities. This additional redundancy makes the α - p CP particularly suitable for applications commonly modeled by the p CP, including emergency service placement and humanitarian relief operations, where robustness of solutions is of critical importance.

The problem can be formally defined as follows: We are given a set of points N , together with a positive integer $p < |N|$ and an integer $\alpha \leq p$. For every pair of points $i, j \in N$, a nonnegative distance d_{ij} is given. A solution is feasible if it selects a subset $P \subseteq N$ of exactly p points to serve as facilities.

For a given feasible solution P , the set of demand points is defined as $N \setminus P$. Hence, the demand set depends on the selected facility locations and consists of all points at which no facility is opened. For any demand point $i \in N \setminus P$, the α -distance $d_\alpha(P, i)$ associated with the solution P is defined as

$$d_\alpha(P, i) = \min_{A \subseteq P, |A|=\alpha} \max_{j \in A} \{d_{ij}\},$$

so the α -distance $d_\alpha(P, i)$ gives the distance of i to the α -closest open facility for the feasible solution P . The objective function value $f_\alpha(P)$ of a feasible solution P is defined as

$$f_\alpha(P) = \max_{i \in N \setminus P} d_\alpha(P, i).$$

Using these definitions, the α - p CP can be formulated as

$$\min_{P \subseteq N, |P|=p} f_\alpha(P).$$

The classical p CP arises as a special case of the problem by setting $\alpha = 1$ and assuming $d_{ii} = 0$ for all $i \in N$. To date, studies on the α - p CP have primarily concentrated on approximation algorithms, including those proposed by Krumke [3], and on heuristic algorithms developed by Mousavi [4] and by Sánchez-Oro, López-Sánchez, Hernández-Díaz, and Duarte [5].

In this talk, we present exact algorithms for solving the α - p CP. Three distinct mixed-integer programming formulations are introduced, together with families of valid inequalities, (iterative) lifting techniques for selected constraints, and inequalities that do not alter the optimal objective value. In addition, we

propose an (iterative) variable-fixing strategy. The lifting methods exploit problem-specific lower bounds and generalize known results for the p CP by Gaar and Sinnl [2]. We show that repeated application of the lifting and variable-fixing procedures drives the formulations toward particular fractional set covering solutions.

Moreover, we prove that the optimal objective value of a semi-relaxed version of one formulation, i.e., where one class of binary variables is retained while the other is relaxed, can be computed in polynomial time via iterative variable fixing. This finding extends earlier results by Elloumi, Labbé, and Pochet [1] for the p CP and a fault-tolerant variant of the p CP. A polyhedral analysis comparing the proposed formulations is also provided.

Based on these formulations and theoretical results, we design branch-and-cut algorithms for the α - p CP. The algorithms integrate both a starting heuristic and a primal heuristic to improve computational performance. Their effectiveness is assessed on benchmark instances from the literature [4, 5]. The computational study shows that our algorithms manage to prove optimality for 116 out of 194 instances. In addition, the best-known objective values are improved for 116 instances. Notably, the 116 instances mentioned in both sentences are not identical, as some instances had previously been solved optimally by heuristics without a corresponding proof of optimality.

References

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