Supportedness in Multi-Objective Minimum Cost Flow Problem: Representation Quality and Output-Sensitive Complexity

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Abstract

This talk addresses the Multi-Objective Integer Minimum Cost Flow (MOIMCF) problem, focusing on the subset of supported nondominated points. While multi-objective combinatorial optimization problems are generally intractable, supported nondominated points offer a more computationally accessible and representative subset of the complete nondominated point set. We present findings from three recent studies demonstrating that supported nondominated points provide superior representation quality compared to extreme supported points, particularly in network problems with increasing arc capacities. Furthermore, we establish an output-polynomial time algorithm for enumerating all supported efficient solutions and discuss the theoretical limits of enumerating supported nondominated vectors, complemented by new, efficient objective space methods.

1 Introduction and Motivation

Network optimization problems, such as the minimum cost flow problem, are fundamental to combinatorial optimization. While various polynomial-time algorithms exist for single-objective variants, real-world scenarios, ranging from transportation and logistics to power distribution networks and supply chains ([1] present over 150 applications), often involve multiple conflicting objectives (e.g., minimizing cost vs. minimizing time vs. maximizing sustainability). These form the class of Multi-Objective Integer Minimum Cost Flow (MOIMCF) problems.

In this context, one is interested in finding solutions with the property that one can only improve with respect to one objective if at least one other objective deteriorates. Such a solution is called an *efficient solution*, and its image is called the *nondominated vector*. In multi-objective combinatorial optimization, the set of all nondominated vectors is typically divided into three categories: extreme nondominated supported ($\mathcal{Y}_{ES}$), supported nondominated ($\mathcal{Y}_S$), and unsupported or weakly supported points.

Determining the complete nondominated set is computationally intractable, in the sense that the number of nondominated points grows exponentially with the problem size [2]. Consequently, practitioners often seek a representative subset. While weakly supported and unsupported points are particularly difficult to compute and generally outnumber supported points ([8]), supported points are more straightforward to determine, as they are optimal solutions to a weighted sum scalarization and often form the basis for the first phase of well known two-phase methods.

This talk focuses on the subset of *supported* nondominated points. We demonstrate that these provide a high-quality representation of the complete nondominated point set and present the first output-polynomial time algorithm to determine the complete set of supported efficient solutions. Moreover, we introduce new objective-space methods for determining the supported nondominated vectors.

2 Representation Quality

A key question in multi-objective optimization is whether a subset of points can adequately represent the complete nondominated point set. A Recent study [7] has indicated that for binary multi-objective combinatorial problems, the set of extreme supported nondominated points ($\mathcal{Y}_{ES}$) already provides a high-quality representation, rendering the computation of non-extreme supported points ($\mathcal{Y}_S \setminus \mathcal{Y}_{ES}$) unnecessary.

However, based on our findings in [5], we demonstrate that this observation does not generalize to network optimization problems with increasing arc capacities. We evaluate the quality of supported nondominated points and extreme supported nondominated points as representations of the complete nondominated point set across various network optimization problems.

Our numerical analysis, utilizing quality metrics such as the Hypervolume Ratio (HVR) and Coverage Error (CE), reveals that $\mathcal{Y}_S$ significantly outperforms $\mathcal{Y}_{ES}$. For instances with higher arc capacities, $\mathcal{Y}_{ES}$ yields insufficient representation quality (significantly lower HVR), whereas $\mathcal{Y}_S$ consistently maintains a near-optimal HVR regardless of problem size or capacity scaling.

These findings establish $\mathcal{Y}_S$ as the superior choice for representing the complete nondominated point set in multi-objective network flow problems.

3 Determination of Supported Solutions

Given the superior representation quality of supported non-diminished points, we address the computational complexity of enumerating them. We distinguish between the decision space (efficient flows, $\mathcal{X}_S$) and the objective space (nondominated vectors, $\mathcal{Y}_S$).

3.1 Output-Polynomial Time Enumeration of $\mathcal{X}_S$

We present an algorithm capable of enumerating all supported efficient *solutions* (flows) for MOIMCF in output-polynomial time [6].

The algorithm operates in two phases: First, the approach determines all extreme supported nondominated vectors and the weighting vectors for each maximally nondominated face. Then, it successively determines all supported efficient solutions in the preimage of each maximally nondominated face by determining all optimal solutions for the corresponding single-objective parametric network flow problem. This is achieved using a modified depth-first search technique that identifies proper zero-cost cycles in the residual graph, called the all optimum flow algorithm recently presented in [4].

3.2 Output-Sensitive Complexity of $\mathcal{Y}_S$

While enumerating the supported efficient solutions can be done in output-polynomial time, determining the supported nondominated *vectors* is more subtle. In [3], we analyze the complexity of determining these vectors.

We show that deciding the existence of a supported nondominated point between two given supported vectors is NP-complete. Consequently, there cannot exist an output-polynomial time algorithm that enumerates supported nondominated vectors in a lexicographically ordered manner unless $P = NP$. This negative result notably contrasts with the positive result for enumerating flows, emphasizing the difficulty introduced by the mapping from decision to objective space, where exponentially many flows may map to a single point.

Despite this, we propose novel methods for identifying supported nondominated points, accompanied by a numerical comparison between decision-space and objective-space methods. The numerical tests highlight that the outcome space methods clearly outperform the decision-space methods, even if they do not run in output-polynomial time. A novel, equivalent, and more compact formulation of the minimum cost flow ILP formulation used in the ε-constraint scalarization approach is introduced and shows a significant time improvement compared to the conventional ILP [3].

4 Conclusion

Supported nondominated points play a major role in MOIMCF. While providing high-quality representations, they are also more straightforward to compute than the unsupported ones and can serve as a foundation for two-phase methods. Unlike unsupported solutions or general efficient flows, they admit output-polynomial time enumeration algorithms in the decision space. These findings provide a theoretical foundation and practical guidelines for decision-makers solving complex network flow problems.

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