Mixed-integer programming approaches for the p-α-closest-center problem

Elisabeth Gaar[1], Sara Joosten[2], and Markus Sinnl[3]

[1]Institute of Mathematics, University of Augsburg, Augsburg, Germany, ✉ elisabeth.gaar@uni-a.de
[2]Institute of Mathematics, University of Augsburg, Augsburg, Germany, ✉ sara.joosten@uni-a.de
[3]Institute of Business Analytics and Technology Transformation/JKU Business School, Johannes Kepler University, Linz, Austria, ✉ markus.sinnl@jku.at

Abstract

Recently, the p-second-center problem was introduced as a variant of a well-known facility location problem, the p-center problem. In this work, we introduce and study the p-α-closest-center problem, which is a generalization of the p-second-center problem, and propose several mixed-integer programming formulations of it. We conduct a polyhedral study and present computational experiments.
Keywords: p-center problem; mixed-integer linear programming; facility location

In 1964, Hakimi [4] introduced the *p-center problem* (pCP), a fundamental facility location problem, in which we are given a set of potential facility locations, a set of customer locations as well as distances between any potential facility location and any customer location. The goal is to open facilities at p of the potential facility locations such that the maximum distance of any customer location to its closest open facility is minimized.

The assumption that open facilities may become unavailable has led to several variants of the pCP such as the *α-neighbor-p-center problem* (αNpCP), introduced by Krumke [5], and the *p-next-center problem* (pNCP), proposed by Albareda-Sambola, Hinojosa, Marín, and Puerto [1]. Recently, Lopéz-Sánchez, Sánchez-Oro, and Hernández-Diaz [6] introduced the *p-second-center problem* (pSCP), in which the aim is to minimize the maximum sum of the distances to the closest and second-closest open facility for any customer location. Ristic, Urošević, Mladenović, and Todosijević [7] proposed a variable neighborhood search heuristic for the pSCP, which is currently the only available solution method for the pSCP in the literature.

In this work, we introduce the *p-α-closest-center problem* ($p\alpha$CCP), which is defined as follows. Let I be the set of customer locations, J be the set of potential facility locations, d_{ij} be the distance between a customer location $i \in I$ and a potential facility location $j \in J$ for each $i \in I$ and $j \in J$, and let α and p be integers with $1 \leq \alpha \leq p < |J|$. Then, the $p\alpha$CCP is defined as

$$\min_{\substack{P \subseteq J \\ |P|=p}} \max_{i \in I} \min_{\substack{A \subseteq P \\ |A|=\alpha}} \sum_{j \in A} d_{ij},$$

i.e., as minimizing the maximum sum of the distances between a customer location and its α closest open facilities for any customer location. With this definition, we obtain the pCP and the pSCP as special cases of the $p\alpha$CCP for $\alpha = 1$ and $\alpha = 2$, respectively. Figure 1 shows an exemplary instance of the $p\alpha$CCP from the TSPlib with Euclidean distances between the locations in $I = J$, which are given as coordinates in the two-dimensional plane, together with an optimal solution of the pCP ($\alpha = 1$) and an optimal solution of the pSCP ($\alpha = 2$) for $p = 5$. As one can see for this instance, the obtained optimal solutions differ. In our work, we compare the optimal objective function values of the same instance for several pCP variants, including the αNpCP, the pNCP, and the $p\alpha$CCP, for varying values of α with each other.

In order to tackle the $p\alpha$CCP, we propose two mixed-integer programming (MIP) formulations of the $p\alpha$CCP, a two-indexed and a subset-indexed formulation, both based on the classical textbook formulation for the pCP (see, e.g., Daskin [2]). In particular, we obtain the first MIP formulation of the pSCP as well

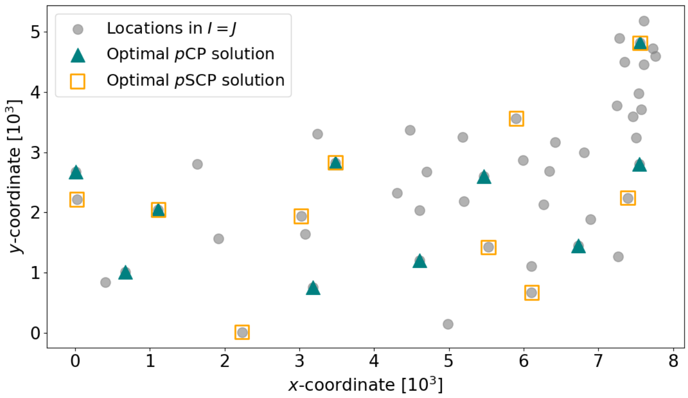

Figure 1: Instance `att48` with an optimal solution for the pCP with objective function value 1203.18 and an optimal solution for the pSCP with objective function value 2827.72.

as the first exact solution approach for the $p\alpha$CCP and the pSCP by using an off-the-shelf MIP-solver. We strengthen our MIP formulations of the $p\alpha$CCP by adding valid inequalities that lift sums of distances smaller than a known lower bound on the optimal objective function value of the $p\alpha$CCP to this lower bound. We also add so-called optimality-preserving inequalities that might cut off some feasible solutions but are valid for at least one optimal solution. These are based on the idea that knowing an upper bound UB on the optimal objective function value of the $p\alpha$CCP we can disregard assignments with sums of distances higher than UB. Similar to the algorithm presented in Gaar and Sinnl [3] for the pCP, we propose an iterative procedure to improve a given lower bound on the optimal objective function value of the $p\alpha$CCP and prove theoretical results on the best lower bound obtainable by this procedure. We also conduct a polyhedral study by proving relations between the linear programming relaxations of our MIP formulations.

Moreover, we perform a computational study on instances from the literature for various values of p and α. We enhance an off-the-shelf branch-and-cut solver with different techniques that decrease computation time by exploiting our theoretical results. For example, we add cutting planes based on our valid and optimality-preserving inequalities. Moreover, we develop a starting and primal heuristic for our problem and introduce separation schemes. We compare our enhanced branch-and-cut algorithm not only with the performance of an off-the-self-solver without any enhancements, but also with heuristic results from the literature. Furthermore, we discuss limitations and possible future research directions.

References

[1] Maria Albareda-Sambola, Yolanda Hinojosa, Alfredo Marín, and Justo Puerto. When centers can fail: A close second opportunity. *Computers & Operations Research*, 62:145–156, 2015.

[2] Mark S Daskin. *Network and discrete location: models, algorithms, and applications*. John Wiley & Sons, 2nd edition, 2013.

[3] Elisabeth Gaar and Markus Sinnl. A scaleable projection-based branch-and-cut algorithm for the p-center problem. *European Journal of Operational Research*, 303(1):78–98, 2022.

[4] Seifollah Louis Hakimi. Optimum distribution of switching centers in a communication network and some related graph theoretic problems. *Operations Research*, 13:462–475, 1965.

[5] Sven Oliver Krumke. On a generalization of the p-center problem. *Information Processing Letters*, 56(2):67–71, 1995.

[6] Ana Dolores López-Sánchez, Jesús Sánchez-Oro, and Alfredo García Hernández-Díaz. GRASP and VNS for solving the p-next center problem. *Computers & Operations Research*, 104:295–303, 2019.

[7] Dalibor Ristic, Dragan Urošević, Nenad Mladenović, and Raca Todosijević. Solving the p-second center problem with variable neighborhood search. *Computer Science and Information Systems*, 20(1):95–115, 2023.