Two-Stage Fixed-Charge Transportation Problem: A Polyhedral Study

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1 Introduction

The *two-stage fixed-charge transportation problem* (TS-FCTP) extends the well-known transshipment problem by also including a fixed charge for shipping any quantity from a source node to a sink node. The TS-FCTP is known to be *NP*-hard problem. To the best of our knowledge, exact solution methods specially tailored to solve TS-FCTP efficiently have not been explored in the literature. In this paper, we propose a cutting-plane based method to solve TS-FCTP, for which we propose two classes of valid inequalities (VIs). We also provide the conditions under which one of the proposed VIs is facet-defining for the projection of the TS-FCTP polyhedron onto the binary space.

2 Mathematical Problem Formulation

TS-FCTP is formally defined in the literature as follows. Let $I, J,$and K denote the sets of source nodes, transshipment nodes, and sink nodes, respectively. Each source node $i \in I$ has a supply capacity $a_i > 0$ while each sink node $k \in K$ has a demand $b_k > 0$. We assume that the problem is balanced, i.e. $\sum\limits_{i\in I} a_i = \sum\limits_{k\in K} b_k = D$. Let C_{ij} and F_{ij} denote the per unit and the fixed shipping costs, respectively, on arc $(i,j), i \in I, j \in J$. T_{jk} and G_{jk} are similarly defined on arc $(j,k), j \in J, k \in K$. We use the decision variables x_{ij} to represent the quantities shipped on arc $(i,j), i \in I, j \in J$, while $z_{ij} = 1$ if arc (i,j) is used, 0 otherwise. The variables y_{jk} and w_{jk} are similarly defined to represent the quantities shipped and the use of the arc $(j,k), j \in J, k \in K$, respectively. The TS-FCTP formulation is:

$$\min \quad \sum_{i\in I}\sum_{j\in J}(F_{ij}z_{ij} + C_{ij}x_{ij}) + \sum_{j\in J}\sum_{k\in K}(G_{jk}w_{jk} + T_{jk}y_{jk}) \tag{1}$$

$$\text{s.t.} \quad \sum_{j\in J} x_{ij} = a_i \qquad \forall\, i \in I \tag{2}$$

$$\sum_{j\in J} y_{jk} = b_k \qquad \forall\, k \in K \tag{3}$$

$$\sum_{i\in I} x_{ij} = \sum_{k\in K} y_{jk} \qquad \forall\, j \in J \tag{4}$$

$$x_{ij} \le a_i z_{ij} \qquad \forall\, i \in I,\, j \in J \tag{5}$$

$$y_{jk} \le b_k w_{jk} \qquad \forall\, j \in J,\, k \in K \tag{6}$$

$$x_{ij} \ge 0,\ z_{ij} \in \{0,1\} \qquad \forall\, i \in I,\, j \in J \tag{7}$$

$$y_{jk} \ge 0,\ w_{jk} \in \{0,1\} \qquad \forall\, j \in J,\, k \in K \tag{8}$$

3 Valid Inequalities for TS-FCTP

For a given set $P \subseteq I$, and a set $Q \subseteq K$, let $A(P) = \sum\limits_{i\in P} a_i$ and $B(Q) = \sum\limits_{k\in Q} b_k$. Let $\overline{P} = I \setminus P$, and $\overline{Q} = K \setminus Q$. The net demand over any subset $P \cup \overline{Q}$ of sources and sinks is defined as $\alpha(P,Q) = A(P) - B(\overline{Q})$.

Let Ω denote the set of (P,Q) subsets such that $\alpha(P,Q) > 0$. The TS-FCTP set is defined as:

$$\text{TS-FCTP} = \left\{ \begin{array}{l} z_{ij} \in \{0,1\}^{|I|\times|J|},\ w_{jk} \in \{0,1\}^{|J|\times|K|} : \\ \displaystyle\sum_{i\in P}\sum_{j\in R} z_{ij} + \sum_{j\in\overline{R}}\sum_{k\in Q} w_{jk} \geq 1, \forall\, P \subseteq I,\ R \subseteq J,\ Q \subseteq K : (P,Q) \in \Omega \end{array} \right\}$$

The TS-FCTP polytope is defined as $\xi = Conv(\text{TS-FCTP})$. We assume that ξ is full-dimensional. Let $E_{PR} = \{(i,j) : i \in P, j \in R\}, E_{\overline{R}Q} = \{(j,k) : j \in \overline{R}, k \in Q\}$ and a similar definition are defines for other arc sets such as $E_{P\overline{R}}, E_{\overline{P}R}$, etc.

$$\xi^{PRQ} = \xi \cap \{z, w : z_{ij} = 1\ \forall\ (i,j) \in E_{P\overline{R}}, E_{\overline{P}R}, E_{\overline{PR}} \text{ and } w_{jk} = 1\ \forall\ (j,k) \in E_{RQ}, E_{R\overline{Q}}, E_{\overline{RQ}}\}$$

Now we propose two classes of VIs for the TS-FCTP. For this, we define $\overline{m}(P,Q) = \max_{i\in P, k\in Q}\{a_i, b_k\}$ and $\underline{m}(P,Q) = \min_{i\in P, k\in Q}\{a_i, b_k\}$.

Theorem 1. *Given $P, \overline{P} \subseteq I,\ R, \overline{R} \subseteq J,$ and $Q, \overline{Q} \subseteq K : (P,Q) \in \Omega$.*

i. The inequality

$$\sum_{i\in P}\sum_{j\in R} z_{ij} + \sum_{j\in\overline{R}}\sum_{k\in Q} w_{jk} \geq 1 \tag{9}$$

is a valid inequality for TS-FCTP. We refer to (9) as a PRQ Type 1 inequality.

ii. The PRQ Type 1 VI is a facet defining inequality for ξ^{PRQ} iff $(P,Q) \in \Omega$ such that $\alpha(P,Q) \leq \underline{m}(P,Q)$.

iii. Let $|R| \geq 2$, $|\overline{R}| \geq 2$. Then, the PRQ Type 1 VI is a facet defining inequality for ξ iff $(P,Q) \in \Omega$ such that $\alpha(P,Q) \leq \underline{m}(P,Q)$.

Theorem 2. *Given $P \subseteq I$, $R \subseteq J$, and $Q \subseteq K$ such that $\alpha(P,Q) > \overline{m}(P,Q)$, the inequality*

$$\sum_{i\in P}\sum_{j\in R} z_{ij} + \sum_{j\in\overline{R}}\sum_{k\in Q} w_{jk} \geq 2 \tag{10}$$

is a valid inequality for the TS-FCTP. We refer to (10) as a PRQ Type 2 inequality.

4 Computational Results

We test the effectiveness of *PRQ Type 1 and Type 2* inequalities, when added at the root node of the branch-and-bound tree, in solving instances of TS-FCTP using the CPLEX solver. The benchmark data set generation scheme has been provided for the FCTP by [1] with a cost parameter θ defined as $\theta = \frac{C_{ij} \times D}{(|I|+|J|-1) \times F_{ij}}$. We extend the same dataset generation procedure for TS-FCTP.

Table 1: Computational results for Dataset 1 with $I = 20, J = 10, K = 30, B = 30$

(a) $\theta = 0.2$

	z_{LP0}	z_{LPC}	$\#T2$	$\#T1$	T_{cplex}	T_C	BB_0	BB_C
Average	94.89%	**96.5%**	73	2	1687.74	**1091.65**	1.05	**0.71**

(b) $\theta = 0$

	z_{LP0}	z_{LPC}	$\#T2$	$\#T1$	T_{cplex}	T_C	BB_0	BB_C
Average	94.84%	**96.46%**	74	2	1212.48	**775.82**	0.75	**0.49**

$\boldsymbol{z_{LP0}}$: LP relaxation as a % of IP optimal; $\boldsymbol{z_{LPC}}$: LP with all PRQ Type 1 and Type 2 cuts as a % of IP optimal; **#T1**, **#T2**: number of PRQ Type 1 and Type 2 cuts; $\boldsymbol{T_{cplex}}$, $\boldsymbol{T_C}$: Computation time (in seconds) without and with PRQ Type 1 and Type 2 cuts; $\boldsymbol{BB_0}$, $\boldsymbol{BB_C}$: branch-and-bound nodes (in millions) without and with PRQ Type 1 and Type 2 cuts.

As clear from Table 1, *PRQ Type 1 and Type 2* cuts help improve the lower bound of TSFCTP by 1.70%, and reduce the size of the branch-and-bound tree by 33%, which helps cut down the CPU time by 35% on average (10 instances in each Table 1 **(a)** and 1 **(b)**).

References

[1] Roberto Roberti, Enrico Bartolini, and Aristide Mingozzi. The fixed charge transportation problem: An exact algorithm based on a new integer programming formulation. *Management Science*, 61(6):1275–1291, 2015.