

Valid Inequalities for a Shortest-Path Routing Optimization Problem

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Abstract

In autonomous systems of the Internet packets are routed on shortest paths to their destinations, for example according to the ECMP principle. The problem of finding a feasible traffic routing configuration realized on paths which can be generated by a system of weights assigned to IP links is NP-hard. This problem can be formulated as a mixed-integer program and attempted with a branch-and-cut algorithm if effective cuts (valid inequalities) can be derived. In this paper we present exact and approximate LP- and MIP-based methods for generating valid inequalities that separate fractional solutions of the basic problem. Besides, a family of complementary valid inequalities, generated with a shortest-path algorithm, related to combinatorial properties of feasible traffic routes is introduced to speed up the cut generation process. Results of a numerical study illustrating computational issues are discussed.

1 Introduction

Traffic engineering (TE) is one of the major tools in short-term network planning. It aims at optimizing the traffic flows in order to provide the required grade of service and meet the assumed network performance objectives. A fundamental instrument of TE is the capability to optimize traffic routing parameters. In circuit-switched networks routing optimization has been recognized as a powerful tool that allows to maximize the amount of traffic handled by a network, and to limit the losses of traffic encountered by individual traffic streams. Recently, extensive research has been devoted to the methods of optimizing traffic routing in the autonomous systems (AS) of the Internet. It has been shown that by optimizing administrative weights of links it is possible to substantially increase network throughput, limit link overloads, minimize packet transmission delays, make a network robust to resource failures, etc., see [1].

Routers of an AS use a routing protocol (like OSPF, IS-IS, etc.) of the Interior-Gateway Protocol (IGP) class to exchange information about the state of the domain's resources [2]. Most of IGP protocols are link-state protocols which allow to distribute information about the state of links, in particular their operational state and administratively assigned values of static parameters (for example routing weights). Newer versions of those protocols allow to distribute TE-related information: link capacity, available bandwidth, packet loss, packet delay, etc. Knowing the topology of the AS and the administrative weights of links, each router is able to compute all shortest paths to every other router of the AS. Having set up its routing tables, the router directs traffic flows to the outgoing links which belong to the shortest paths to particular traffic destinations. If there are several such links for a particular destination, the traffic to that destination is split equally among those links according to the ECMP (equal-cost multi-path) principle .

The problem of finding a feasible traffic routing configuration realized on paths generated by a system of weights assigned to IP links is NP-hard [3]. This problem can be formulated as a mixed-integer program and attempted with a branch-and-cut algorithm if effective cuts (valid inequalities) can be derived. In this paper we first present a formulation of a basic routing problem for an AS that uses the OSPF/ECMP type of routing (Section 2), formulate a condition for feasibility of routing variables (Section 3). Then, we present exact and approximate LP- and MIP-based methods for generating valid inequalities that separate fractional solutions of the linear relaxations of the basic problem (Section 4). Next, we introduce a family of complementary valid inequalities, which are related to combinatorial properties of feasible traffic routes, and can be generated by a shortest-path algorithm (Section 5). Finally, results of a numerical study illustrating the influence of the cuts on the branch-and-cut process (Section 6).

2 Routing problem formulation

An AS is modeled by means of a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, with the set of nodes \mathcal{V} and the set of links \mathcal{E} . The originating and terminating node of link $e \in \mathcal{E}$ is denoted by $a(e)$ and $b(e)$, respectively. Then, $\delta^+(v)$ and $\delta^-(v)$ are the sets of links originating and terminating, respectively, at node $v \in \mathcal{V}$, i.e., $\delta^+(v) = \{e \in \mathcal{E} : a(e) = v\}$ and $\delta^-(v) = \{e \in \mathcal{E} : b(e) = v\}$. Capacities of links are given and denoted by c_e , $e \in \mathcal{E}$. The volume of traffic generated in node $v \in \mathcal{V}$ and destined to node $t \in \mathcal{V}$ is also given and denoted by h_{vt} .

Suppose that $\mathbf{x} = (x_{et} \geq 0 : e \in \mathcal{E}, t \in \mathcal{V})$, $\mathbf{u} = (u_{et} \in \{0, 1\} : e \in \mathcal{E}, t \in \mathcal{V})$, $\mathbf{z} = (z_{vt} \geq 0 : v, t \in \mathcal{V})$, $\mathbf{r} = (r_{vt} \geq 0 : v, t \in \mathcal{V})$, $\mathbf{w} = (w_e \geq 1 : e \in \mathcal{E})$ are variables. Then the basic routing design problem is formulated as follows (see [3], [4], [5]):

$$\mathbf{P}: \max f(\mathbf{x}, \mathbf{u}) \tag{1a}$$

$$\text{s.t.} \quad \sum_{e \in \delta^-(t)} x_{et} = \sum_{v \in \mathcal{V} \setminus \{t\}} h_{vt} \quad t \in \mathcal{V} \tag{1b}$$

$$\sum_{e \in \delta^+(v)} x_{et} - \sum_{e \in \delta^-(v)} x_{et} = h_{vt} \quad v, t \in \mathcal{V}, v \neq t \tag{1c}$$

$$\sum_{t \in \mathcal{V} \setminus \{a(e)\}} x_{et} \leq c_e \quad e \in \mathcal{E} \tag{1d}$$

$$0 \leq x_{et} \leq u_{et} \sum_{v \in \mathcal{V} \setminus \{t\}} h_{vt} \quad e \in \mathcal{V}, t \in \mathcal{V} \tag{1e}$$

$$0 \leq z_{a(e)t} - x_{et} \leq (1 - u_{et}) \sum_{v \in \mathcal{V} \setminus \{t\}} h_{vt} \quad e \in \mathcal{E}, t \in \mathcal{V} \tag{1f}$$

$$r_{b(e)v} + w_e - r_{a(e)v} + u_{ev} \geq 1 \quad e \in \mathcal{E}, v \in \mathcal{V}, \tag{1g}$$

$$r_{b(e)v} + w_e - r_{a(e)v} \leq M(1 - u_{ev}) \quad e \in \mathcal{E}, v \in \mathcal{V}, \tag{1h}$$

$$w_e \geq 1 \quad e \in \mathcal{E}, \tag{1i}$$

$$r_{vv} = 0 \quad v \in \mathcal{V}. \tag{1j}$$

Subproblem (1a)-(1d) is a node-link formulation of a capacitated multicommodity flow allocation problem with aggregated flows \mathbf{x} , where x_{et} denotes the total traffic destined to node t carried on link e ($f(\mathbf{x}, \mathbf{u})$ is an assumed routing design objective). The next two constraints express the rules of traffic routing by means of binary routing variables \mathbf{u} . Constraints (1e) force that traffic destined to node t can only use links allowed by the routing configuration ($u_{et} = 1$), while constraint (1f) ensures that in each node traffic is split equally among links assigned for its destination. For node $v \in \mathcal{V}$ and destination $t \in \mathcal{V}$ this common value of equal split is expressed by variable z_{vt} . Finally, shortest-path routing constraints (1g)-(1j) assure that the routing vector \mathbf{u} defines shortest paths consistent with the weight system \mathbf{w} . Variable r_{vt} is supposed to express the distance between nodes (length of the shortest path with respect to \mathbf{w}) from node v to node t . The quantity $r_{b(e)v} + w_e - r_{a(e)v}$ measures the difference between the length of the shortest path which starts in node $a(e)$, goes over link e and ends in node v , and the distance from the starting node of e to v . Thus, link e is on a shortest path to node v if, and only if, $r_{b(e)v} + w_e - r_{a(e)v} = 0$.

Problem (1) is NP-hard [3], [1], [6]. Its formulation (and similar formulations given in [7], [8], [9]) constitute a MIP problem, and as such can be solved using methods of integer programming, in particular the branch-and-bound (B&B). However, B&B turns out not to be effective in this case because of the presence of "big M " in constraint (1h). Therefore the formulation of the problem should be strengthened with carefully selected valid inequalities, and the problem attempted with the branch-and-cut (B&C) approach.

In the balance of this paper \mathcal{U} will denote the set of all admissible binary routing vectors \mathbf{u} , i.e., $\mathbf{u} \in \mathcal{U}$ if, and only if, there exist weights $\mathbf{w} \geq \mathbf{1}$ and distances $\mathbf{r} \geq \mathbf{0}$ such that constraints (1g)-(1j) are satisfied. Whether problem (1) can be solved effectively by integer programming methods depends on efficient representation of the set \mathcal{U} of admissible routing configurations, stronger that the representation involving big M . This issue is analyzed in the next sections.

3 Routing feasibility

The set \mathcal{U} of admissible routing vectors is the set of all binary vectors \mathbf{u} that satisfy constraints (1g)-(1j). In Section 2 of the companion paper [10] a linear programming problem is formulated that can be used for testing whether a given binary vector \mathbf{u} is admissible or not; the problem is based on constraints (1g)-(1j) but, since \mathbf{u} is given, the condition $(r_{b(e)v} + w_e - r_{a(e)v}) \cdot u_{ev} = 0$, $e \in \mathcal{E}, v \in \mathcal{V}$ is used instead of (1h). A dual formulation of that problem is analyzed that uses the vectors $\boldsymbol{\mu} = (\mu_{ev} : e \in \mathcal{E}, v \in \mathcal{V})$, $\boldsymbol{\pi} = (\pi_{ev} : e \in \mathcal{E}, v \in \mathcal{V})$ and $\boldsymbol{\theta} = (\theta_e : e \in \mathcal{E})$ of the dual variables, corresponding to constraints (1g), (1h) and (1i), respectively. Using some algebra, after eliminating variables θ_e , and substituting variables μ_{ev} with new variables $\varphi_{ev} = u_{ev}\pi_{ev} - \mu_{ev}$ (variables $\boldsymbol{\varphi}$ may be interpreted as flow variables as explained below), the following form of the dual problem (expressed in variables $\boldsymbol{\pi} = (\pi_{ev} : e \in \mathcal{E}, v \in \mathcal{V})$ and $\boldsymbol{\varphi} = (\varphi_{ev} : e \in \mathcal{E}, v \in \mathcal{V})$) is obtained:

$$\mathbf{F}(\mathbf{u}): \quad \max F_{\mathbf{u}}(\boldsymbol{\varphi}, \boldsymbol{\pi}) = \sum_{e \in \mathcal{E}} \sum_{v \in \mathcal{V}} u_{ev} \varphi_{ev} + \sum_{e \in \mathcal{E}} \sum_{v \in \mathcal{V}} (1 - u_{ev}) u_{ev} \pi_{ev} \quad (2a)$$

$$\text{s.t.} \quad \sum_{v \in \mathcal{V}} \varphi_{ev} \geq 0 \quad e \in \mathcal{E} \quad (2b)$$

$$\sum_{e \in \delta^+(t)} \varphi_{ev} = \sum_{e \in \delta^-(t)} \varphi_{ev} \quad v, t \in \mathcal{V} \quad (2c)$$

$$\sum_{e \in \mathcal{E}} \sum_{v \in \mathcal{V}} \varphi_{ev} \geq \sum_{e \in \mathcal{E}} \sum_{v \in \mathcal{V}} u_{ev} \pi_{ev} - (1 - \sum_{e \in \mathcal{E}} \sum_{v \in \mathcal{V}} \pi_{ev}) \quad (2d)$$

$$\varphi_{ev} \leq u_{ev} \pi_{ev}, \quad \pi_{ev} \geq 0 \quad e \in \mathcal{E}, v \in \mathcal{V}. \quad (2e)$$

The routing vector admissibility condition reads: $\mathbf{u} \in \mathcal{U}$ if, and only if, $F_{\mathbf{u}}^* = 0$ (where $F_{\mathbf{u}}^*$ denotes the optimal objective of (2)). This condition is valid for binary and non-binary continuous \mathbf{u} with $0 \leq u_{et} \leq 1$, $e \in \mathcal{E}, t \in \mathcal{V}$. It can be shown that for any \mathbf{u} , binary or not, $0 \leq F_{\mathbf{u}}^* \leq 1$. Notice that the zero vector $\boldsymbol{\varphi} = \mathbf{0}$ together with $\hat{\boldsymbol{\pi}}$ such that $\hat{\pi}_{ev} \equiv \alpha = (2 \cdot |\mathcal{E}| \cdot |\mathcal{V}|)^{-1}$ is a feasible solution of problem $\mathbf{F}(\mathbf{u})$ for any \mathbf{u} . Therefore $F_{\mathbf{u}}^* \geq \alpha \sum_{e \in \mathcal{E}} \sum_{v \in \mathcal{V}} (1 - u_{ev}) u_{ev}$. Certainly, the last expression is always strictly positive for fractional vectors \mathbf{u} (with $0 < u_{et} < 1$ for at least one pair (e, t)) which means that fractional \mathbf{u} 's are always non-admissible as routing configurations. On the other hand, for binary vectors \mathbf{u} this expression is always equal to 0 so in the binary case the objective function reduces to $F_{\mathbf{u}}(\boldsymbol{\varphi}) = \sum_{e \in \mathcal{E}} \sum_{v \in \mathcal{V}} u_{ev} \varphi_{ev}$. Finally, problem (2) can be compactly stated as

$$\max_{(\boldsymbol{\varphi}, \boldsymbol{\pi}) \in \mathcal{F}_{\mathbf{u}}} F_{\mathbf{u}}(\boldsymbol{\varphi}, \boldsymbol{\pi}), \quad (3)$$

where $\mathcal{F}_{\mathbf{u}}$ is the solution space of (2), i.e., the set of all pairs $(\boldsymbol{\varphi}, \boldsymbol{\pi})$ satisfying constraints (2b)-(2e). In fact $\mathcal{F}_{\mathbf{u}}$ defines a class of circular flows $\boldsymbol{\varphi}$; the interpretation of (2) in terms of circular flows is discussed in [10].

4 Valid inequalities

In this section we consider the problem of separating infeasible shortest-path routing vectors \mathbf{u} through appropriate valid inequalities (cuts) derived from problem (2). We start with the following lemma.

Lemma 4.1. *Let $(\boldsymbol{\varphi}^*, \boldsymbol{\pi}^*) \in \mathcal{F}_{\mathbf{u}^0}$ for some binary or fractional vector \mathbf{u}^0 , and let \mathbf{u} be a binary vector with $u_{ev} = 1$ whenever $\varphi_{ev}^* > 0$. Then there exists vector $\boldsymbol{\pi} = (\pi_{ev} : (e, v) \in \mathcal{E} \times \mathcal{V})$ such that $(\boldsymbol{\varphi}^*, \boldsymbol{\pi}) \in \mathcal{F}_{\mathbf{u}}$.*

Proof. Let $\mathcal{I} = \{(e, v) : \varphi_{ev}^* > 0\}$. Define vector $\boldsymbol{\pi}$ as follows: $\pi_{ev} = u_{ev}^0 \pi_{ev}^*$ for $(e, v) \in \mathcal{I}$, and $\pi_{ev} = 0$ otherwise. We must show that constraints (2d) and (2e) are satisfied. Constraint (2e) is satisfied because for $\varphi_{ev}^* > 0$ we have $\varphi_{ev}^* \leq u_{ev}^0 \pi_{ev}^* = \pi_{ev} = u_{ev} \pi_{ev}$ and for $\varphi_{ev}^* \leq 0$ it is satisfied for any π_{ev} including $\pi_{ev} = 0$. Constraint (2d) is satisfied because $\sum_{e \in \mathcal{E}} \sum_{v \in \mathcal{V}} (1 + u_{ev}^0) \pi_{ev}^* = \sum_{(e,v) \in \mathcal{I}} (1 + u_{ev}^0) \pi_{ev}^* + \sum_{(e,v) \in (\mathcal{E} \times \mathcal{V}) \setminus \mathcal{I}} (1 + u_{ev}^0) \pi_{ev}^* = \sum_{(e,v) \in \mathcal{I}} u_{ev}^0 \pi_{ev}^* + \sum_{(e,v) \in \mathcal{I}} \pi_{ev}^* + \sum_{(e,v) \in (\mathcal{E} \times \mathcal{V}) \setminus \mathcal{I}} (1 + u_{ev}^0) \pi_{ev}^* = \sum_{(e,v) \in \mathcal{I}} u_{ev} \pi_{ev} + \sum_{(e,v) \in \mathcal{I}} \pi_{ev}^* + \sum_{(e,v) \in (\mathcal{E} \times \mathcal{V}) \setminus \mathcal{I}} (1 + u_{ev}^0) \pi_{ev}^* \geq \sum_{(e,v) \in \mathcal{I}} u_{ev} \pi_{ev} + \sum_{(e,v) \in \mathcal{I}} \pi_{ev} = \sum_{e \in \mathcal{E}} \sum_{v \in \mathcal{V}} (1 + u_{ev}) \pi_{ev}$. \square

Now suppose that \mathcal{I}^1 and \mathcal{I}^0 are two disjoint sets of pairs $(e, v) \in \mathcal{E} \times \mathcal{V}$, i.e., $\mathcal{I}^1, \mathcal{I}^0 \subseteq \mathcal{E} \times \mathcal{V}$ and $\mathcal{I}^1 \cap \mathcal{I}^0 = \emptyset$. Let S denote a real-valued function defined as follows:

$$S(\mathcal{I}^1, \mathcal{I}^0; \mathbf{u}) = \sum_{(e,v) \in \mathcal{I}^1} (1 - u_{ev}) + \sum_{(e,v) \in \mathcal{I}^0} u_{ev}. \quad (4)$$

We will consider valid inequalities of the following form:

$$S(\mathcal{I}^1, \mathcal{I}^0; \mathbf{u}) \geq 1. \quad (5)$$

For binary \mathbf{u}^0 all terms in (4) are binary, so $S(\mathcal{I}^1, \mathcal{I}^0; \mathbf{u}^0)$ is non-negative and integer. If (5) is not satisfied then $S(\mathcal{I}^1, \mathcal{I}^0; \mathbf{u}^0) = 0$, and all its terms must be equal to 0. Hence, the following remark holds.

Remark 4.1. Inequality (5) does not hold for a binary vector \mathbf{u}^0 (i.e., $S(\mathcal{I}^1, \mathcal{I}^0; \mathbf{u}^0) = 0$) if, and only, if $u_{ev}^0 = 1$ for all $(e, v) \in \mathcal{I}^1$ and $u_{ev}^0 = 0$ for all $(e, v) \in \mathcal{I}^0$.

The following proposition shows how to separate a non-admissible binary vector $\mathbf{u}^0 \notin \mathcal{U}$.

Proposition 4.2. *Suppose \mathbf{u}^0 is a binary vector defining a non-admissible routing configuration, and (φ, π) is a solution of problem $\mathbf{F}(\mathbf{u}^0)$ such that $F_{\mathbf{u}^0}(\varphi) > 0$. Then inequality (5) with $\mathcal{I}^1 = \{(e, v) \in \mathcal{E} \times \mathcal{V} : u_{ev}^0 = 1 \wedge \varphi_{ev} \neq 0\}$ and $\mathcal{I}^0 = \{(e, v) \in \mathcal{E} \times \mathcal{V} : u_{ev}^0 = 0 \wedge \varphi_{ev} \neq 0\}$ separates \mathbf{u}^0 and does not separate any feasible routing vector \mathbf{u} (i.e., any binary $\mathbf{u} \in \mathcal{U}$).*

Proof. Due to the definition of sets \mathcal{I}^1 and \mathcal{I}^0 , it holds that $S(\mathcal{I}^1, \mathcal{I}^0; \mathbf{u}^0) = 0$, and hence \mathbf{u}^0 is separated. Now, consider any vector \mathbf{u} that is also separated by inequality (5) with $S(\mathcal{I}^1, \mathcal{I}^0; \mathbf{u}) \geq 1$. Due to Remark 4.1, if $\varphi_{ev} \neq 0$ then $u_{ev}^0 = u_{ev}$. Thus, (φ, π) is also a feasible solution of $\mathbf{F}(\mathbf{u})$, i.e., of problem (2) specified for vector \mathbf{u} . Hence, $F_{\mathbf{u}}(\varphi) = F_{\mathbf{u}^0}(\varphi) > 0$ and \mathbf{u} is non-admissible. \square

The next proposition demonstrates that it is possible to enforce the cut by defining a smaller set \mathcal{I}^1 .

Proposition 4.3. *Suppose \mathbf{u}^0 is a binary vector defining a non-admissible routing configuration, and (φ, π) is a solution of problem $\mathbf{F}(\mathbf{u}^0)$ such that $F_{\mathbf{u}^0}(\varphi) > 0$. Then inequality (5) with $\mathcal{I}^1 = \mathcal{I}_+^1(\mathbf{u}^0) = \{(e, v) \in \mathcal{E} \times \mathcal{V} : u_{ev}^0 = 1 \wedge \varphi_{ev} > 0\}$ and $\mathcal{I}^0 = \mathcal{I}_-^0(\mathbf{u}^0) = \{(e, v) \in \mathcal{E} \times \mathcal{V} : u_{ev}^0 = 0 \wedge \varphi_{ev} < 0\}$ separates \mathbf{u}^0 and does not separate any feasible routing vector \mathbf{u} (i.e., any binary $\mathbf{u} \in \mathcal{U}$).*

Proof. Due to the definition of sets \mathcal{I}^1 and \mathcal{I}^0 , it holds that $S(\mathcal{I}^1, \mathcal{I}^0; \mathbf{u}^0) = 0$, and hence \mathbf{u}^0 is separated. Now, consider any vector \mathbf{u} that is also separated by inequality $S(\mathcal{I}^1, \mathcal{I}^0; \mathbf{u}) \geq 1$. Notice that if $\varphi_{ev} \neq 0$ then $(e, v) \in \mathcal{I}_+^1(\mathbf{u}^0) \cup \mathcal{I}_-^0(\mathbf{u}^0) \cup \mathcal{I}_-^1(\mathbf{u}^0)$, where $\mathcal{I}_-^1(\mathbf{u}^0) = \{(e, v) \in \mathcal{E} \times \mathcal{V} : u_{ev}^0 = 1 \wedge \varphi_{ev} < 0\}$. Due to Remark 4.1 and the definition of set \mathcal{I}^1 , according to Lemma 4.1 there exists vector $\hat{\pi}$ such that $(\varphi, \hat{\pi}) \in \mathcal{F}_{\mathbf{u}}$. Obviously, $u_{ev} \leq u_{ev}^0$ if $(e, v) \in \mathcal{I}_-^1(\mathbf{u}^0)$, and, due to Remark 4.1, $u_{ev} = u_{ev}^0$ if $(e, v) \in \mathcal{I}_+^1(\mathbf{u}^0) \cup \mathcal{I}_-^0(\mathbf{u}^0)$. Hence, $u_{ev}\varphi_{ev} \geq u_{ev}^0\varphi_{ev}$ and $F_{\mathbf{u}}(\varphi) \geq F_{\mathbf{u}^0}(\varphi) > 0$. Thus, \mathbf{u} is non-admissible. \square

Separating a fractional vector \mathbf{u}^0 (recall that \mathbf{u} with at least one strictly fractional u_{ev} always describes an infeasible shortest-path routing configuration) is more complex. The goal is to find sets \mathcal{I}^1 and \mathcal{I}^0 for which inequality (5) separates \mathbf{u}^0 and does not separate any admissible routing configuration, and that these sets determine the most violated valid inequality of the considered type. Let $\mathbf{q} = (q_{ev} : e \in \mathcal{E}, v \in \mathcal{V})$ be a binary vector and define $\mathcal{J}(\mathbf{q}) = \{(e, v) \in \mathcal{E} \times \mathcal{V} : q_{ev} = 1\}$ (i.e., \mathbf{q} is the characteristic function of set $\mathcal{J}(\mathbf{q})$). Suppose that $\mathcal{I}^1 = \mathcal{J}(\mathbf{y})$ and $\mathcal{I}^0 = \mathcal{J}(\mathbf{z})$ for two binary vectors $\mathbf{y} = (y_{ev} : e \in \mathcal{E}, v \in \mathcal{V})$ and $\mathbf{z} = (z_{ev} : e \in \mathcal{E}, v \in \mathcal{V})$. We assume that sets \mathcal{I}^0 and \mathcal{I}^1 are disjoint, so $y_{ev} + z_{ev} \leq 1$ must hold for all pairs (e, v) . Then function (4) can be calculated as:

$$S(\mathcal{I}^1, \mathcal{I}^0; \mathbf{u}) = \sum_{e \in \mathcal{E}} \sum_{v \in \mathcal{V}} ((1 - u_{ev})y_{ev} + u_{ev}z_{ev}). \quad (6)$$

Now, for an infeasible \mathbf{u}^0 , an issue arises of how to determine such vectors \mathbf{y} and \mathbf{z} that $S(\mathcal{J}(\mathbf{y}), \mathcal{J}(\mathbf{z}); \mathbf{u}^0) < 1$ and all binary vectors \mathbf{u} separated by (5) correspond to infeasible routing configurations. Our attempt is as

follows. Consider the following problem (where Δ is a small strictly positive constant):

$$\mathbf{G}(\mathbf{u}): \quad \min G_{\mathbf{u}}(\mathbf{y}, \mathbf{z}) = \sum_{e \in \mathcal{E}} \sum_{v \in \mathcal{V}} ((1 - u_{ev})y_{ev} + u_{ev}z_{ev}) \quad (7a)$$

$$\text{s.t. } (\boldsymbol{\varphi}, \boldsymbol{\pi}) \in \mathcal{F}_{\mathbf{u}} \quad (7b)$$

$$\sum_{e \in \mathcal{E}} \sum_{v \in \mathcal{V}} u_{ev} \varphi_{ev} \geq \Delta \quad (7c)$$

$$\varphi_{ev} \leq y_{ev} \quad e \in \mathcal{E}, v \in \mathcal{V} \quad (7d)$$

$$-\varphi_{ev} \leq y_{ev} + z_{ev} \quad e \in \mathcal{E}, v \in \mathcal{V} \quad (7e)$$

$$y_{ev} + z_{ev} \leq 1 \quad e \in \mathcal{E}, v \in \mathcal{V} \quad (7f)$$

$$y_{ev}, z_{ev} \in \{0, 1\} \quad e \in \mathcal{E}, v \in \mathcal{V}. \quad (7g)$$

Let problem $\mathbf{G}(\mathbf{u}^0)$ be feasible for some \mathbf{u}^0 ; denote its optimal solution by $(\boldsymbol{\varphi}^*(\mathbf{u}^0), \boldsymbol{\pi}^*(\mathbf{u}^0), \mathbf{y}^*(\mathbf{u}^0), \mathbf{z}^*(\mathbf{u}^0))$, and by $G_{\mathbf{u}^0}^*$ —the optimal value of the objective function (when it does not lead to confusion, short notation $\boldsymbol{\varphi}^*, \boldsymbol{\pi}^*, \mathbf{y}^*, \mathbf{z}^*$ and G^* , respectively, will be used). The following properties hold: $(\boldsymbol{\varphi}^*, \boldsymbol{\pi}^*)$ is a feasible solution of problem $\mathbf{F}(\mathbf{u}^0)$ (due to (7b)); $F_{\mathbf{u}}(\boldsymbol{\varphi}^*, \boldsymbol{\pi}^*) > 0$ (due to (7c)); if $\varphi_{ev}^* > 0$ then $y_{ev}^* = 1$ (due to (7d)); if $\varphi_{ev}^* < 0$ then $y_{ev}^* = 1$ or $z_{ev}^* = 1$ (due to (7e)). The role of problem $\mathbf{G}(\mathbf{u})$ in separating fractional infeasible shortest-path routing configurations is as follows.

Proposition 4.4. *Let \mathbf{u}^0 be a (fractional) routing vector and suppose that problem $\mathbf{G}(\mathbf{u}^0)$ is feasible and that $G^* = G_{\mathbf{u}^0}^* < 1$. Then inequality (5) with $\mathcal{I}^1 = \mathcal{J}(\mathbf{y}^*(\mathbf{u}^0))$ and $\mathcal{I}^0 = \mathcal{J}(\mathbf{z}^*(\mathbf{u}^0))$ separates \mathbf{u}^0 , and it does not separate any feasible binary shortest-path routing configuration vector $\mathbf{u} \in \mathcal{U}$.*

Proof. Let $(\boldsymbol{\varphi}^*, \boldsymbol{\pi}^*, \mathbf{y}^*, \mathbf{z}^*)$ be an optimal solution of $\mathbf{G}(\mathbf{u}^0)$. Due to the assumed form of the objective function (7a) it holds that $S(\mathcal{I}^1, \mathcal{I}^0; \mathbf{u}^0) = G^* < 1$, which means that \mathbf{u}^0 is separated by (5).

Now, let \mathbf{u} be a binary vector separated by (5) which means, due to Remark 4.1, that if $(e, v) \in \mathcal{I}^1$ then $u_{ev} = 1$, and if $(e, v) \in \mathcal{I}^0$ then $u_{ev} = 0$. We will demonstrate that $\mathbf{u} \notin \mathcal{U}$. Due to the definition of the set \mathcal{I}^1 , according to Lemma 4.1 there exists vector $\boldsymbol{\pi}$ such that $(\boldsymbol{\varphi}^*, \boldsymbol{\pi}) \in \mathcal{F}_{\mathbf{u}}$. Now consider the value of the objective of problem $\mathbf{F}(\mathbf{u})$ for $\boldsymbol{\varphi}^*$: $F_{\mathbf{u}}(\boldsymbol{\varphi}^*) = \sum_{e \in \mathcal{E}} \sum_{v \in \mathcal{V}} u_{ev} \varphi_{ev}^*$. The definition of sets \mathcal{I}^1 and \mathcal{I}^0 together with constraints (7d) and (7e) imply that if $(e, v) \notin \mathcal{I}^1 \cup \mathcal{I}^0$ then $y_{ev}^* = z_{ev}^* = 0$ and $\varphi_{ev}^* = 0$. Thus, $F_{\mathbf{u}}(\boldsymbol{\varphi}^*) = \sum_{(e,v) \in \mathcal{I}^1} u_{ev} \varphi_{ev}^* + \sum_{(e,v) \in \mathcal{I}^0} u_{ev} \varphi_{ev}^* = \sum_{(e,v) \in \mathcal{I}^1} \varphi_{ev}^* \geq \sum_{(e,v) \in \mathcal{I}^1} u_{ev}^0 \varphi_{ev}^*$. Further, we note that $\sum_{(e,v) \in \mathcal{I}^1} u_{ev}^0 \varphi_{ev}^* = \sum_{e \in \mathcal{E}} \sum_{v \in \mathcal{V}} u_{ev}^0 \varphi_{ev}^* - \sum_{(e,v) \in \mathcal{I}^0} u_{ev}^0 \varphi_{ev}^*$. Now, if $(e, v) \notin \mathcal{I}^1$ then $y_{ev}^* = 0$, and due to (7d), if $y_{ev}^* = 0$ then $\varphi_{ev}^* \leq 0$. Thus, $\sum_{(e,v) \in \mathcal{I}^0} u_{ev}^0 \varphi_{ev}^*$ is non-positive and using (7c) we finally get $F_{\mathbf{u}}(\boldsymbol{\varphi}^*) \geq \sum_{e \in \mathcal{E}} \sum_{v \in \mathcal{V}} u_{ev} \varphi_{ev}^* \geq \Delta > 0$, which means that \mathbf{u} is a non-admissible routing vector. \square

Again, the cut can be made stronger by defining a smaller set \mathcal{I}^1 . Consider a modification of $\mathbf{G}(\mathbf{u})$:

$$\mathbf{H}(\mathbf{u}): \quad \min H_{\mathbf{u}}(\mathbf{y}, \mathbf{z}) = \sum_{e \in \mathcal{E}} \sum_{v \in \mathcal{V}} ((1 - u_{ev})y_{ev} + u_{ev}z_{ev}) \quad (8a)$$

$$\text{s.t. } (\boldsymbol{\varphi}, \boldsymbol{\pi}) \in \mathcal{F}_{\mathbf{u}} \quad (8b)$$

$$\sum_{e \in \mathcal{E}} \sum_{v \in \mathcal{V}} \alpha_{ev} - \sum_{e \in \mathcal{E}} \sum_{v \in \mathcal{V}} \beta_{ev} \geq \Delta \quad (8c)$$

$$\varphi_{ev} \leq \alpha_{ev} \leq \varphi_{ev} + 1 - y_{ev} \quad e \in \mathcal{E}, v \in \mathcal{V} \quad (8d)$$

$$-\varphi_{ev} \leq \beta_{ev} + \gamma_{ev} \leq -\varphi_{ev} + 1 - (x_{ev} + z_{ev}) \quad e \in \mathcal{E}, v \in \mathcal{V} \quad (8e)$$

$$y_{ev} \geq \alpha_{ev}, \quad x_{ev} \geq \beta_{ev}, \quad z_{ev} \geq \gamma_{ev} \quad e \in \mathcal{E}, v \in \mathcal{V} \quad (8f)$$

$$y_{ev} + x_{ev} + z_{ev} \leq 1 \quad e \in \mathcal{E}, v \in \mathcal{V} \quad (8g)$$

$$\alpha_{ev} \geq 0, \quad \beta_{ev} \geq 0, \quad \gamma_{ev} \geq 0 \quad e \in \mathcal{E}, v \in \mathcal{V} \quad (8h)$$

$$y_{ev}, x_{ev}, z_{ev} \in \{0, 1\} \quad e \in \mathcal{E}, v \in \mathcal{V}. \quad (8i)$$

(Above, variables x_{ev} are local and have nothing to do with the flow variables in problem (1).) Let problem $\mathbf{H}(\mathbf{u}^0)$ be feasible for some \mathbf{u}^0 ; denote the vectors of its optimal solution by $\boldsymbol{\varphi}^*(\mathbf{u}^0), \boldsymbol{\pi}^*(\mathbf{u}^0), \mathbf{y}^*(\mathbf{u}^0)$,

$\alpha^*(\mathbf{u}^0)$, etc., and by $H_{\mathbf{u}^0}^*$ —the optimal value of the objective function. The following properties hold: (φ^*, π^*) is a feasible solution of problem $\mathbf{F}(\mathbf{u}^0)$ (due to (8b)); $F_{\mathbf{u}}(\varphi^*, \pi^*) > 0$ (due to (8c)); if $\varphi_{ev}^* > 0$ then $\alpha_{ev}^* = \varphi_{ev}^*$ and $y_{ev}^* = 1$ (due to (8d) and (8f)); if $\varphi_{ev}^* < 0$ then either $\beta_{ev}^* = -\varphi_{ev}^*$ and $x_{ev}^* = 1$ or $\gamma_{ev}^* = -\varphi_{ev}^*$ and $z_{ev}^* = 1$ (due to (8e) and (8f)).

Proposition 4.5. *Let \mathbf{u}^0 be a (fractional) routing vector and suppose that problem $\mathbf{H}(\mathbf{u}^0)$ is feasible and that $H^* = H_{\mathbf{u}^0}^* < 1$. Then inequality (5) with $\mathcal{I}^1 = \mathcal{J}(\mathbf{y}^*(\mathbf{u}^0))$ and $\mathcal{I}^0 = \mathcal{J}(\mathbf{z}^*(\mathbf{u}^0))$ separates \mathbf{u}^0 , and it does not separate any feasible binary shortest-path routing configuration vector $\mathbf{u} \in \mathcal{U}$.*

Proof. Let $(\varphi^*, \pi^*, \mathbf{y}^*, \mathbf{z}^*)$ be an optimal solution of $\mathbf{H}(\mathbf{u}^0)$. Due to the assumed form of the objective function (7a) it holds that $S(\mathcal{I}^1, \mathcal{I}^0; \mathbf{u}^0) = H^* < 1$, which means that \mathbf{u}^0 is separated by (5).

Now, let \mathbf{u} be a binary vector separated by (5) which means, due to Remark 4.1, that if $(e, v) \in \mathcal{I}^1$ then $u_{ev} = 1$, and if $(e, v) \in \mathcal{I}^0$ then $u_{ev} = 0$. We will demonstrate that $\mathbf{u} \notin \mathcal{U}$. Due to the definition of the set \mathcal{I}^1 , according to Lemma 4.1 there exists vector π such that $(\varphi^*, \pi) \in \mathcal{F}_{\mathbf{u}}$. Consider the value of the objective of problem $\mathbf{F}(\mathbf{u})$ for φ^* : $F_{\mathbf{u}}(\varphi^*) = \sum_{e \in \mathcal{E}} \sum_{v \in \mathcal{V}} u_{ev} \varphi_{ev}^*$. Define $\mathcal{I}_+^1 = \{(e, v) \in \mathcal{E} \times \mathcal{V} : y_{ev}^* = 1 \wedge \varphi_{ev}^* > 0\}$, $\mathcal{I}_-^1 = \{(e, v) \in \mathcal{E} \times \mathcal{V} : x_{ev}^* = 1 \wedge \varphi_{ev}^* < 0\}$ and $\mathcal{I}_-^0 = \{(e, v) \in \mathcal{E} \times \mathcal{V} : z_{ev}^* = 1 \wedge \varphi_{ev}^* < 0\}$; thus if $\varphi_{ev}^* \neq 0$ then $(e, v) \in \mathcal{I}_+^1 \cup \mathcal{I}_-^1 \cup \mathcal{I}_-^0$. Since $\mathcal{I}^1 = \mathcal{I}_+^1$ and $\mathcal{I}^0 = \mathcal{I}_-^0$, thus $\sum_{e \in \mathcal{E}} \sum_{v \in \mathcal{V}} u_{ev} \varphi_{ev}^* = \sum_{(e,v) \in \mathcal{I}_+^1} u_{ev} \alpha_{ev}^* - \sum_{(e,v) \in \mathcal{I}_-^1} u_{ev} \beta_{ev}^* - \sum_{(e,v) \in \mathcal{I}_-^0} u_{ev} \gamma_{ev}^* = \sum_{(e,v) \in \mathcal{I}_+^1} \alpha_{ev}^* - \sum_{(e,v) \in \mathcal{I}_-^1} u_{ev} \beta_{ev}^* \geq \sum_{(e,v) \in \mathcal{I}_+^1} \alpha_{ev}^* - \sum_{(e,v) \in \mathcal{I}_-^0} \beta_{ev}^* = \sum_{e \in \mathcal{E}} \sum_{v \in \mathcal{V}} \alpha_{ev}^* - \sum_{e \in \mathcal{E}} \sum_{v \in \mathcal{V}} \beta_{ev}^* \geq \Delta > 0$, which means that \mathbf{u} defines a non-admissible routing configuration. \square

Since the minimum of the objective function of problem $\mathbf{H}(\mathbf{u})$ is at the same time the minimum of expression $S(\mathcal{I}^1, \mathcal{I}^0; \mathbf{u})$, if the problem is feasible then its optimal solution determines the most violated valid inequality. Naturally, this holds only if $H_{\mathbf{u}}^* < 1$, otherwise no violated inequality of the considered type exists.

However, in contrast to the LP problem $\mathbf{F}(\mathbf{u})$, both problems $\mathbf{G}(\mathbf{u})$ and $\mathbf{H}(\mathbf{u})$ are MIPs. Trying to separate fractional vectors \mathbf{u} more effectively, we consider a linear relaxation of problem $\mathbf{G}(\mathbf{u})$. Consider a fixed pair (e, v) . The two corresponding variables y_{ev} and z_{ev} ($0 \leq y_{ev}, z_{ev} \leq 1$) are related to variable φ_{ev} only through constraints (7d) and (7e), and are non-negative. Since objective function (7a) is minimized and all its coefficients are non-negative, in any optimal solution at least one of the two variables y_{ev} and z_{ev} is equal to 0. More precisely, $\varphi_{ev}^* > 0$ implies $y_{ev}^* \geq \varphi_{ev}^*$ and $z_{ev}^* = 0$. If $\varphi_{ev}^* < 0$, then $(1 - u_{ev}) < u_{ev}$ implies $y_{ev}^* \geq -\varphi_{ev}^*$ and $z_{ev}^* = 0$; otherwise, $z_{ev}^* \geq -\varphi_{ev}^*$ and $y_{ev}^* = 0$. (Note: constraint (7f) which is used to make sets \mathcal{I}^1 and \mathcal{I}^0 disjoint can thus be skipped.) This implies that for the linear relaxation of problem (7), out of the two values of y_{ev}^* and z_{ev}^* , one is equal to $|\varphi_{ev}^*|$ and the other to 0. Consequently, the definition of $\mathcal{J}(\mathbf{q})$ can be modified to cover fractional vectors \mathbf{q} : $\mathcal{J}(\mathbf{q}) = \{(e, v) \in \mathcal{E} \times \mathcal{V} : q_{ev} > 0\}$, because sets $\mathcal{J}(\mathbf{y}^*)$ and $\mathcal{J}(\mathbf{z}^*)$ are disjoint. Then, inequality (5) is still properly defined in the sense that admissible routing configurations are not separated. This is because the proof of the second part of Proposition 4.4 does not rely on the fact that y_{ev}^* is equal to 1 but rather on that it is greater than 0.

Unfortunately, $G_{\mathbf{u}}^*$ is in general not equal to $S(\mathcal{J}(\mathbf{y}^*), \mathcal{J}(\mathbf{z}^*); \mathbf{u})$. However, one may still use an approximate separation procedure that consists in solving the linear relaxation and evaluating the resulting value of $S(\mathcal{J}(\mathbf{y}^*), \mathcal{J}(\mathbf{z}^*); \mathbf{u})$: a valid inequality is found if $S(\mathcal{J}(\mathbf{y}^*), \mathcal{J}(\mathbf{z}^*); \mathbf{u}) < 1$. It should be noted that due to the fact that variables φ_{ev} describe circular flows, and constraint (2b) requires that negative flows are compensated by positive flows, in practice usually all non-zero values φ_{ev}^* are equal to Δ or $-\Delta$, for some $0 < \Delta \leq 1$. When this is the case, then $G_{\mathbf{u}}^* = \Delta \cdot S(\mathcal{J}(\mathbf{y}^*), \mathcal{J}(\mathbf{z}^*); \mathbf{u})$, and a valid inequality separating \mathbf{u} is thus found if $G_{\mathbf{u}}^* < \Delta$.

5 Combinatorial cuts

In this section we will introduce three properties, and corresponding valid inequalities of the form (5), based directly on the properties of shortest paths. They will be called: transit, split, and cycle (see Figure 5). We note that other types of such combinatorial cuts have been proposed in [9], [11], [12], [13], [14], [15], [16].

The *transit* property expresses a relation between shortest paths to a destination and shortest paths to the transit nodes of those paths to the destination. Assume that there is a path from node s to node t (such path

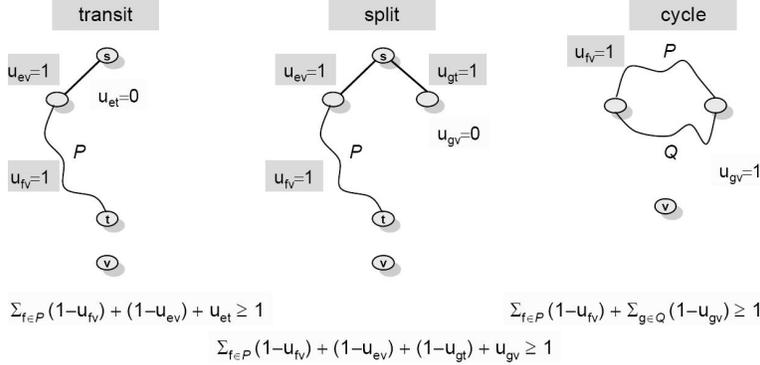


Figure 1: Types of combinatorial valid inequalities

can be decomposed into the starting link e and path $\mathcal{P}_{b(e)t}$ from $b(e)$ to t with all links belonging to shortest paths to some node v . Thus, t is a transit node on a shortest path from s to v . Then all links between s and t on the path to v , in particular link e , must belong to a shortest path to t , as if there were a shorter path from s to t then it should be used to v as well. The following inequality separates vectors \mathbf{u} which contradict this property:

$$\sum_{f \in \mathcal{P}_{b(e)t}} (1 - u_{fv}) + (1 - u_{ev}) + u_{et} \geq 1. \quad (9)$$

To find the most violated inequality (9) for given s, v, t , and e , it is enough to find a shortest path $\mathcal{P}_{b(e)t}$ from $b(e)$ to t using the values $1 - u_{ev}$ for the link weights. This inequality is stronger than the one used in [9], because it skips the values of the variables defining shortest paths from t to v , and considers the path from s to t and not from s to v .

The *split* property expresses a relation between splitting traffic among shortest paths to a destination and among shortest paths to the transit nodes. Assume the same situation as in the first case, and, additionally, that there is another link g originating in node s which is on a shortest path to node t . Then g must be on a shortest path to v , because there are two paths of equal length from s to t (one starting with link g and one with link e because of the *transit* property), and as one is used to reach v , the other should be used to reach v as well. The vectors \mathbf{u} that contradict this property can be separated using the following inequality:

$$\sum_{f \in \mathcal{P}_{b(e)t}} (1 - u_{fv}) + (1 - u_{ev}) + (1 - u_{gt}) + u_{gv} \geq 1. \quad (10)$$

To find the most violated inequality (10) for given s, v, t, e , and g it is enough to find the same path $\mathcal{P}_{b(e)t}$ as in the case of the *transit* property.

The *cycle* property expresses a relation between shortest paths to a single destination. Since the values of link weights are strictly positive, on a shortest path to a destination the distances of the consecutive nodes to that destination are decreasing. Thus, the segments \mathcal{P}_{st} and \mathcal{Q}_{ts} of two such paths cannot form a cycle. The following inequality separates vectors \mathbf{u} which contradict this property:

$$\sum_{f \in \mathcal{P}_{st}} (1 - u_{fv}) + \sum_{g \in \mathcal{Q}_{ts}} (1 - u_{gv}) \geq 1. \quad (11)$$

To find the most violated inequality (11) for given s, t , and v , it is enough to find a pair of shortest paths, from s to t , and from t to s , respectively, using the values $1 - u_{ev}$ as the link weights.

To find all violated inequalities of types (9)-(11) for a given vector \mathbf{u} , it is thus sufficient to determine, for each destination node v , the shortest paths between all pairs of nodes, using values $1 - u_{ev}$ as link weights; thus, the entire process has overall complexity of $O(|\mathcal{V}|^4)$.

6 Numerical experiments

For the purpose of numerical studies we have programmed problem (1) using the Callable Library API of CPLEX 10.1. The problem is solved by means of the CPLEX's built-in B&C solver calling an external cut-generation procedure every time a B&C node relaxation has been solved. The procedure uses a fractional solution obtained in a B&C node and searches for violated valid inequalities and combinatorial cuts discussed in the paper. If a violated cut is found the relaxation is re-optimized.

The goal of the numerical experiments was to examine the influence of the introduced cuts on the branch-and-cut process. We used a 6-node network with 28 links (with capacities ranging from 24 to 76) and all possible 30 demands. As an objective function we used $f(\mathbf{x}) = \min_{e \in \mathcal{E}} (c_e - \sum_{t \in \mathcal{V} \setminus \{a(e)\}} x_{et})$, modifying problem (1) accordingly. In our example the optimal objective is equal to 6, while the upper bound resulting from the linear relaxation of (1) is equal to 13.

The cuts were generated using the following methods: CC - combinatorial cuts; MIP(G) - cuts from $\mathbf{G}(\mathbf{u})$; MIP'(G) - cuts from $\mathbf{G}(\mathbf{u})$, with the CPLEX parameters set to: depth-first-search, objective's upper bound, emphasize optimal solutions, limited number of required incumbents; LR(G) - cuts from linear relaxation of $\mathbf{G}(\mathbf{u})$; MIP(H) - cuts from $\mathbf{H}(\mathbf{u})$ (with time limit); CPLEX - only internal CPLEX cuts. Several strategies based on combinations of these options were applied as indicated in column "Cut generation" of Table 1. Each entry of the column indicates the sequence of cut generation methods applied in a B&C node until the first cut is found (if any); symbol || means that the two sequences given as its arguments are applied in parallel. An option was to start the B&C process with an initial set of cuts which had been generated by another scenario; the number of such a scenario is given in column "After".

The scenarios were compared with respect to the computation time, the number of visited B&C nodes, the number of generated CPLEX cuts, and user cuts; the obtained results are presented in the respective columns of Table 1. One more important indicator is the B&C node processing time, i.e., the time that is spent at a single B&C node; it can be estimated as the proportion of the total computation time and the number of visited B&C nodes. Asterisks in columns "Time" and "B&C nodes" mean that the computation, respectively, was stopped due to the imposed time limit or ran out of memory.

Scenario	After	Cut generation	Time [s]	B&C nodes	CPLEX cuts	User cuts
1	-	CPLEX	>4155	>664000*	382	-
2	-	MIP(G)	>41800*	>28000	?	>8000
3	-	LR(G)/MIP(G)	>72700*	>66600	?	>10000
4	-	CC	548	68700	100	1556
5	-	CC/LR(G)	4081	64000	87	2485
6	-	CC/MIP(G)	4200	24600	99	2137
7	-	CC/LR(G)/MIP(G)	13299	24900	95	2785
8	-	CC/LR(G)/MIP'(G)	4071	14500	89	1961
9	-	CC/MIP(H)/LR(G)/MIP(G)	9100	10400	98	1901
10	-	CC/(MIP(H) LR(G)/MIP(G))	8600	8960	81	2030
11	4	CPLEX	292	46400	89	-
12	9	CPLEX	264	21600	85	-
13	10	CPLEX	41	5800	74	-

Table 1: Results of B&C for a 6-node network

As can be seen from the results for scenario 1, the standard B&C process of CPLEX is highly ineffective: the number of B&C nodes is enormous (especially for such a small network).

Scenario 4 shows that with CC this number can be reduced by one order of magnitude maintaining similar reduction of the computation time; generating the combinatorial cuts is quite efficient and does not increase the B&C node processing time much.

As seen from the results for scenario 5, LR(G) does not provide significant gains as far as the number of B&C nodes is concerned; instead, it causes a major increase of the total computation time due to a much larger B&C node processing time, even if LR(G) is combined with CC (notice that the combinatorial cuts are a special case of cuts that are generated by means of LR(G)).

Further significant reduction of the number of B&C nodes can only be obtained when the MIP-generated cuts are used. This is, however, achieved at the high computational cost (this is the reason for introducing time limit in MIP(H)). Usage of the MIP-generated cuts must be combined with CC, otherwise, as proved by scenario 2, the B&C node processing time is excessive. For scenarios 9 and 10, the number of visited B&C nodes is reduced almost by two orders of magnitude in comparison with scenario 1. The results for scenarios 12 and 13 provide another confirmation of the fact that the quality of the MIP-generated cuts, especially the ones provided by solving $\mathbf{H}(\mathbf{u})$, is high.

We wish to note that although the size of the examined network is not impressive, it is used for the problem involving the ECMP type of shortest-path routing, and not for a single-shortest-path routing problem. The former problem is known to be much more difficult than the latter in the computational practice.

7 Final remarks

In this working paper we have discussed new types of valid inequalities for the shortest-path routing optimization problem (1). The inequalities are generated either by using the MIP formulations related to propositions 4.4 and 4.5, or as combinatorial cuts related to the shortest path properties. The presented numerical results reveal that these inequalities can be useful in resolving problem (1) due to reducing the number of B&C tree nodes. Still, more effort has to be undertaken to increase their efficiency. This, in fact, is a direction of our current research carried on in cooperation with France Telecom Research.

We have also studied admissibility of binary routing vectors \mathbf{u} . In propositions 4.2 and 4.3 we have presented a way for generating valid inequalities for unfeasible binary \mathbf{u} . Such inequalities can be used in the two-phase approach for solving shortest-path routing problems. The approach, described for example in Section 7.4 of [3], solves the flow allocation MIP problem (1a)-(1f) in phase 1, and then, in phase 2, checks whether the resulting binary routing vector \mathbf{u}^* is admissible. If it is not, valid inequalities for variables \mathbf{u} (based on propositions 4.2 and 4.3) are added to the problem of phase 1, and the procedure is repeated. Certainly, if applied for current \mathbf{u}^* , combinatorial cuts presented in Section 5 can be used for generating valid inequalities for phase 1 as well.

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