

# Feasibility Issues in Shortest-Path Routing with Traffic Flow Split

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## Abstract

In the Internet's autonomous systems packets are routed on shortest paths to their destinations. A related problem is how to find an admissible traffic routing configuration using paths that can be generated by a system of weights assigned to IP links. This problem is NP-hard. It can be formulated as a mixed-integer program and attempted with a branch-and-cut algorithm if effective cuts (valid inequalities) can be derived. In this paper we discuss admissibility of shortest-path routing configurations represented by binary variables specifying whether or not a particular link is on a shortest path to a particular destination. We present a linear programming problem for testing routing admissibility and derive solutions of this problem which characterize non-admissible routing configurations.

## 1 Introduction

Shortest-path, destination based traffic routing is a widely accepted solution for Internet autonomous systems (AS), supported by such protocols as OSPF and IS-IS [1], [2]. With such routing all packets with a certain destination router (node) that arrive at an originating/intermediate node are forwarded to an outgoing link which is on a shortest path to the destination node. The length of the path is determined with respect to a certain system of administrative link weights which is known to the routers. In the case when there are more (than one) shortest paths to the destination, the packets are equally distributed among all outgoing links that belong to the shortest paths to the considered destination.

Finding administrative weights which optimize traffic flows in an AS leads in general to mixed-integer programming problems [3], [4], [5], [6], [7], which are NP-hard [2], [8]. One of the most promising exact resolution approaches for such problems is to use branch-and-cut (B&C). For this, however, we need to eliminate non-admissible routing configurations in the computational process through introducing appropriate cuts in the nodes of the branch-and-bound tree.

In this paper we discuss admissibility of shortest-path routing configurations represented by binary variables that specify whether or not a particular link is on a shortest path to a particular destination. We demonstrate how to identify non-admissible shortest-path routing configurations by means of a linear program, and derive solutions of this problem which characterize non-admissible routing configurations (sections 3 and 4). As discussed in final remarks (Section 5), the linear relaxation of the problem studied in this paper can be used for deriving cuts for solving a fundamental shortest-path routing optimization problem through B&C. This issue is studied in detail in the companion paper [9].

The idea underlying the considerations of this paper was presented in [10]. In this context our main contribution is two-fold. First, we adjust the dual formulation presented [10] with a special objective function that allows for generating cuts applicable for the B&C approach. Second, we present an original set of non-admissible routing pattern examples revealing that the relation between the (intuitive) degree of non-admissibility of a routing pattern and the optimal value of the corresponding objective function (i.e., the optimal objective value of the dual problem corresponding to the given routing pattern) is not at all obvious.

## 2 Problem formulation

Shortest-path routing of the OSPF/ECMP type (see [1]) can be modeled as follows. An AS is represented by a directed graph  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ , with the set of nodes  $\mathcal{V}$  and the set of links  $\mathcal{E}$ . Let  $a(e)$  and  $b(e)$  denote the originating and terminating node, respectively, of link  $e \in \mathcal{E}$ . Now, let  $\mathbf{u} = (u_{ev} : e \in \mathcal{E}, v \in \mathcal{V})$  be a vector of binary *routing variables* which are supposed to specify a configuration of shortest-path, destination-based routing, i.e.,  $u_{ev}$  equals 1 if, and only if, link  $e \in \mathcal{E}$  is used in node  $a(e)$  to carry traffic with destination  $v \in \mathcal{V}$ . Such a vector  $\mathbf{u}$  is *admissible* (i.e., it defines an admissible shortest-path routing configuration) if the corresponding paths in the network graph can be realized by some system of positive (administrative) link weights  $\mathbf{w} = (w_e : e \in \mathcal{E})$ . More precisely, let  $\mathcal{G}_v = (\mathcal{V}, \mathcal{E}_v)$ , where  $\mathcal{E}_v = \{e \in \mathcal{E} : u_{ev} = 1\}$ , be a subgraph (called an *in-graph*) of  $\mathcal{G}$  for a given node  $v \in \mathcal{V}$ . Then a binary vector  $\mathbf{u}$  is admissible if there exists a system  $\mathbf{w}$  of positive link weights such that for each node  $v \in \mathcal{V}$  the following property holds.

**Property:** For each node  $s \in \mathcal{V} \setminus \{v\}$  any path in subgraph  $\mathcal{G}_v$  from  $s$  to  $v$  is the shortest path with respect to  $\mathbf{w}$ , and all other paths from  $s$  to  $v$  in graph  $\mathcal{G}$  are strictly longer than the paths in  $\mathcal{G}_v$ .

The set of all admissible binary vectors  $\mathbf{u}$  will be denoted by  $\mathcal{U}$ . In the companion paper [9] we explain how routing variables  $\mathbf{u}$  can be linked to destination-based traffic flows to specify an overall shortest-path routing optimization problem. In the present paper we concentrate on the admissibility issues of binary vectors  $\mathbf{u}$ , while a branch-and-cut approach for the overall problem, based on the considerations of this paper, is presented in [9].

Whether or not a routing vector  $\mathbf{u}$  is in set  $\mathcal{U}$  can be determined by a linear programming problem which will be described below. Let  $\mathbf{w}$  be a system of positive link weights, and let  $r_{sv}$  denote the resulting distance from node  $s \in \mathcal{V}$  to node  $v \in \mathcal{V}$ , i.e.,  $r_{sv}$  is the length of a shortest path (with respect to  $\mathbf{w}$ ) from  $s$  to  $v$ . For each  $e \in \mathcal{E}$  and  $v \in \mathcal{V}$  consider the quantity  $\delta_{ev} = r_{b(e)v} + w_e - r_{a(e)v}$ . Clearly,  $\delta_{ev}$  measures the difference between the length of the shortest path which starts in node  $a(e)$ , goes over link  $e$  and terminates in node  $v$ , and the distance from the starting node of  $e$  to  $v$ . Thus, link  $e$  is on a shortest path to node  $v$  if, and only if,  $\delta_{ev} = 0$ . Hence, a routing vector  $\mathbf{u}$  defines a shortest-path routing configuration if there exists a system  $\mathbf{w}$  of positive weights such that:

$$\delta_{ev} \geq 1 \text{ if } u_{ev} = 0 \text{ and } \delta_{ev} = 0 \text{ if } u_{ev} = 1. \quad (1)$$

Observe that the weight system  $\mathbf{w}$  appearing in conditions (1) must have the property that if two paths differ in lengths, then they differ by at least 1. This property is clearly present for positive integer weight systems. For non-integer positive weights this is not a serious restriction, as the property can be assured for any weight system by multiplying all its components by an appropriate positive number ( $w_e \rightarrow \alpha w_e$ ,  $\alpha > 0$ ). The above conditions are well known in the shortest-path theory (see [11]), and have been used in the shortest-path routing context for example in [12] and [10]. For our further purposes we rewrite condition (1) as follows:

$$\delta_{ev} + u_{ev} \geq 1, \quad \delta_{ev} u_{ev} = 0 \quad e \in \mathcal{E}, v \in \mathcal{V} \quad (2)$$

Note that equality  $\delta_{ev} u_{ev} = 0$  in (2) is non-linear in the applications when  $\mathbf{u}$  are variables, as in problem (1) in [9]. Certainly, for the purpose of the integer programming methods, in particular for branch-and-bound, this constraint could be linearized using a big positive constant  $M$ :  $\delta_{ev} \leq M(1 - u_{ev})$ ,  $e \in \mathcal{E}$ ,  $v \in \mathcal{V}$ . Such linearizing, however, can appear not to be effective in terms of the lower bounds in linear relaxations with respect to  $\mathbf{u}$  just because of the presence of the "big  $M$ ".

Thus,  $\mathbf{u} \in \mathcal{U}$  if, and only if,  $\mathbf{u}$  is a binary vector for which there exists a system of positive weights  $\mathbf{w}$  such that conditions (2) are satisfied. It follows that the following linear program in variables  $y$ ,  $\mathbf{w} = (w_e : e \in \mathcal{E})$

and  $\mathbf{r} = (r_{sv} : s, v \in \mathcal{V})$  can be used to check whether a given vector  $\mathbf{u}$  defines admissible routing:

$$\mathbf{P}(\mathbf{u}): \quad \min y \quad (3a)$$

$$\text{s.t. } r_{b(e)v} + w_e - r_{a(e)v} + u_{ev} \geq 1 - y \quad e \in \mathcal{E}, v \in \mathcal{V}, \quad (3b)$$

$$(r_{b(e)v} + w_e - r_{a(e)v})u_{ev} \leq y \quad e \in \mathcal{E}, v \in \mathcal{V}, \quad (3c)$$

$$w_e \geq 1 \quad e \in \mathcal{E}, \quad (3d)$$

$$r_{vv} = 0 \quad v \in \mathcal{V}, \quad (3e)$$

$$y \geq 0. \quad (3f)$$

Note that  $y = 1, \mathbf{r} = \mathbf{0}, \mathbf{w} = \mathbf{1}$  is a feasible solution of  $\mathbf{P}(\mathbf{u})$  for any binary  $\mathbf{u}$ , so the problem is feasible (and, due to (3f), bounded). Let  $(\mathbf{w}^*(\mathbf{u}), \mathbf{r}^*(\mathbf{u}), y^*(\mathbf{u}))$  denote an optimal solution of  $\mathbf{P}(\mathbf{u})$ . If  $y^*(\mathbf{u}) = 0$  then  $\mathbf{w}^*(\mathbf{u})$  and  $\mathbf{r}^*(\mathbf{u})$  satisfy constraints (2), and hence  $\mathbf{u} \in \mathcal{U}$ , i.e.,  $\mathbf{u}$  describes a shortest-path routing configuration. On the other hand, if  $y^*(\mathbf{u}) > 0$ , there is no assignment of link weights which can generate the routing configuration  $\mathbf{u}$ . Thus, set  $\mathcal{U}$  can also be defined as:  $\mathbf{u} \in \mathcal{U}$  if, and only if,  $\mathbf{u}$  is binary and  $y^*(\mathbf{u}) = 0$ . Note also that since  $y = 1, \mathbf{r} = \mathbf{0}, \mathbf{w} = \mathbf{1}$  is a feasible solution for any  $\mathbf{u}$ , in any case  $y^*(\mathbf{u}) \leq 1$ .

Now, consider the problem dual to  $\mathbf{P}(\mathbf{u})$ . Let  $\boldsymbol{\mu} = (\mu_{ev} : e \in \mathcal{E}, v \in \mathcal{V})$ ,  $\boldsymbol{\pi} = (\pi_{ev} : e \in \mathcal{E}, v \in \mathcal{V})$  and  $\boldsymbol{\theta} = (\theta_e : e \in \mathcal{E})$  be the vectors of the dual variables corresponding to constraints (3b), (3c) and (3d), respectively. The dual problem is as follows:

$$\max \sum_{e \in \mathcal{E}} \sum_{v \in \mathcal{V}} \mu_{ev}(1 - u_{ev}) + \sum_{e \in \mathcal{E}} \theta_e \quad (4a)$$

$$\text{s.t. } \sum_{v \in \mathcal{V}} (\mu_{ev} - u_{ev}\pi_{ev}) + \theta_e = 0 \quad e \in \mathcal{E} \quad (4b)$$

$$\sum_{e \in B(t)} (\mu_{ev} - u_{ev}\pi_{ev}) + \sum_{e \in A(t)} (-\mu_{ev} + u_{ev}\pi_{ev}) = 0 \quad v, t \in \mathcal{V} \quad (4c)$$

$$\sum_{e \in \mathcal{E}} \sum_{v \in \mathcal{V}} \mu_{ev} + \sum_{e \in \mathcal{E}} \sum_{v \in \mathcal{V}} \pi_{ev} \leq 1 \quad (4d)$$

$$\mu_{ev} \geq 0, \quad \pi_{ev} \geq 0 \quad e \in \mathcal{E}, v \in \mathcal{V} \quad (4e)$$

$$\theta_e \geq 0 \quad e \in \mathcal{E}. \quad (4f)$$

Using equality (4b), we eliminate dual variables  $\theta_e$ , and after some algebra we arrive at the following form of the dual problem.

$$\mathbf{D}(\mathbf{u}): \quad \max D_{\mathbf{u}}(\boldsymbol{\mu}, \boldsymbol{\pi}) = \sum_{e \in \mathcal{E}} \sum_{v \in \mathcal{V}} u_{ev}(\pi_{ev} - \mu_{ev}) \quad (5a)$$

$$\text{s.t. } \sum_{v \in \mathcal{V}} (u_{ev}\pi_{ev} - \mu_{ev}) \geq 0 \quad e \in \mathcal{E} \quad (5b)$$

$$\sum_{e \in A(t)} (u_{ev}\pi_{ev} - \mu_{ev}) = \sum_{e \in B(t)} (u_{ev}\pi_{ev} - \mu_{ev}) \quad v, t \in \mathcal{V} \quad (5c)$$

$$\sum_{e \in \mathcal{E}} \sum_{v \in \mathcal{V}} \mu_{ev} + \sum_{e \in \mathcal{E}} \sum_{v \in \mathcal{V}} \pi_{ev} \leq 1 \quad (5d)$$

$$\mu_{ev} \geq 0, \quad \pi_{ev} \geq 0 \quad e \in \mathcal{E}, v \in \mathcal{V}. \quad (5e)$$

The primal problem  $\mathbf{P}(\mathbf{u})$  is feasible and bounded for all  $\mathbf{u}$ , hence so is the dual. Let  $(\boldsymbol{\mu}^*(\mathbf{u}), \boldsymbol{\pi}^*(\mathbf{u}))$  denote an optimal solution of  $\mathbf{D}(\mathbf{u})$ , and let  $D_{\mathbf{u}}^* = D_{\mathbf{u}}(\boldsymbol{\mu}^*(\mathbf{u}), \boldsymbol{\pi}^*(\mathbf{u}))$ . Since  $D_{\mathbf{u}} = y^*(\mathbf{u})$ , we have that  $0 \leq D_{\mathbf{u}}^* \leq 1$ , and  $\mathbf{u} \in \mathcal{U}$  if, and only if,  $\mathbf{u}$  is binary and  $D_{\mathbf{u}}^* = 0$ . Now we rewrite objective (5a) as

$$D_{\mathbf{u}}(\boldsymbol{\mu}, \boldsymbol{\pi}) = \sum_{e \in \mathcal{E}} \sum_{v \in \mathcal{V}} u_{ev}(u_{ev}\pi_{ev} - \mu_{ev}) + \sum_{e \in \mathcal{E}} \sum_{v \in \mathcal{V}} (1 - u_{ev})u_{ev}\pi_{ev} \quad (6)$$

and notice that since  $\mathbf{u}$  is binary, the second term in (6) is always equal to 0. Further, we substitute variables  $\mu_{ev}$  with new variables  $\varphi_{ev} = u_{ev}\pi_{ev} - \mu_{ev}$  and obtain a problem equivalent to (5), expressed in variables

$\varphi = (\varphi_{ev} : e \in \mathcal{E}, v \in \mathcal{V})$  and  $\pi = (\pi_{ev} : e \in \mathcal{E}, v \in \mathcal{V})$ :

$$\mathbf{F}(\mathbf{u}): \quad \max F_{\mathbf{u}}(\varphi) = \sum_{e \in \mathcal{E}} \sum_{v \in \mathcal{V}} u_{ev} \varphi_{ev} \quad (7a)$$

$$\text{s.t.} \quad \sum_{v \in \mathcal{V}} \varphi_{ev} \geq 0 \quad e \in \mathcal{E} \quad (7b)$$

$$\sum_{e \in A(t)} \varphi_{ev} = \sum_{e \in B(t)} \varphi_{ev} \quad v, t \in \mathcal{V} \quad (7c)$$

$$\sum_{e \in \mathcal{E}} \sum_{v \in \mathcal{V}} \varphi_{ev} \geq \sum_{e \in \mathcal{E}} \sum_{v \in \mathcal{V}} u_{ev} \pi_{ev} - (1 - \sum_{e \in \mathcal{E}} \sum_{v \in \mathcal{V}} \pi_{ev}) \quad (7d)$$

$$\varphi_{ev} \leq u_{ev} \pi_{ev}, \quad \pi_{ev} \geq 0 \quad e \in \mathcal{E}, v \in \mathcal{V}. \quad (7e)$$

Let  $F_{\mathbf{u}}^*$  denote the optimal objective of (7). Since problems  $\mathbf{D}(\mathbf{u})$  and  $\mathbf{F}(\mathbf{u})$  are equivalent, therefore  $F_{\mathbf{u}}^* = D_{\mathbf{u}}^*$ . Hence, a vector  $\mathbf{u}$  defines an admissible shortest-path routing configuration if, and only if,  $F_{\mathbf{u}}^* = 0$ . Note that the zero vectors  $\varphi = \mathbf{0}$  and  $\pi = \mathbf{0}$  compose a (trivial) feasible solution of problem  $\mathbf{F}(\mathbf{u})$  for any  $\mathbf{u}$ . Thus, for a fixed binary  $\mathbf{u}$ , problem  $\mathbf{F}(\mathbf{u})$  can be regarded as a special type of a multicommodity flow problem with  $\varphi_{ev}$  interpreted as an (bounded, and possibly negative) amount of (pseudo-) flow of commodity  $v$  on link  $e$ . Due to constraint (7c) the flow of each commodity is circular, and due to (7b) the total amount of flow on each link is non-negative. The objective is to find the network flow with maximum total revenue where  $u_{ev}$  is the unit revenue (1 or 0) of using link  $e$  by commodity  $v$ .

### 3 Directed cycles and valid cycles

Consider a candidate binary routing vector  $\mathbf{u}$  and for each node  $v \in \mathcal{V}$  define  $\mathcal{I}(v) = \{e \in \mathcal{E} : u_{ev} = 1\}$ . Let  $\mathcal{C} \subseteq \mathcal{E}$  be a directed cycle in the considered graph  $\mathcal{G}$ . Now suppose that there exists a node  $v$  such that  $\mathcal{C} \subseteq \mathcal{I}(v)$  (observe that this makes vector  $\mathbf{u}$  not admissible, see [11]) and define vectors  $\varphi^*$  and  $\pi^*$  as follows:  $\varphi_{ev}^* = \pi_{ev}^* = \frac{1}{|\mathcal{C}|}$ ,  $e \in \mathcal{C}$ ,  $\varphi_{ev}^* = \pi_{ev}^* = 0$  for all other  $(e, v) \in \mathcal{E} \times \mathcal{V}$ . Then  $(\varphi^*, \pi^*)$  is a feasible solution of (7), as all the constraints are satisfied (in particular constraint (7d)). Moreover,  $F_{\mathbf{u}}(\varphi^*) = 1$  which implies that  $(\varphi^*, \pi^*)$  is an optimal solution of  $\mathbf{F}(\mathbf{u})$ . Moreover, if cycle  $\mathcal{C}$  is simple then the solution is also a vertex-solution.

The problem of whether a given set of arborescences (for the notion of arborescence see [13]) can be uniquely generated by a system of weights was considered by Broström and Holmberg in [10]. They derived a dual formulation analogous to formulation (7) and showed that in most cases a set of arborescences can be shown infeasible using the so called *valid cycles*. The result of [10] translated to our (slightly more general) model using the fixed binary routing vector  $\mathbf{u}$  is as follows. Let  $\mathcal{C} = \mathcal{F} \cup \mathcal{B}$  be a cycle in graph  $\mathcal{G}$ , where  $\mathcal{F} \cap \mathcal{B} = \emptyset$ , and  $\mathcal{F}$  is the set of forward links while  $\mathcal{B}$  is the set of backward links in cycle  $\mathcal{C}$ . Now, let  $s$  and  $t$  be two different nodes. Cycle  $\mathcal{C}$  is called  $(s, t)$ -feasible if  $\mathcal{F} \subseteq \mathcal{I}(s)$  and  $\mathcal{B} \subseteq \mathcal{I}(t)$ . An  $(s, t)$ -feasible cycle  $\mathcal{C}$  is called  $(s, t)$ -valid if  $(\mathcal{B} \setminus \mathcal{I}(s)) \cup (\mathcal{F} \setminus \mathcal{I}(t)) \neq \emptyset$ .

Let  $\Delta = \frac{1}{2|\mathcal{C}|}$  and define flow  $\varphi^*$  and multipliers  $\pi^*$  as follows:  $\varphi_{es}^* = \pi_{es}^* = \Delta$  for  $e \in \mathcal{F}$ ,  $\varphi_{es}^* = -\Delta$  for  $e \in \mathcal{B}$ ;  $\varphi_{et}^* = -\Delta$  for  $e \in \mathcal{F}$ ,  $\varphi_{et}^* = \pi_{et}^* = \Delta$  for  $e \in \mathcal{B}$ ; and all other  $\varphi_{ev}^*$  and  $\pi_{ev}^*$  are equal to 0. We observe that if  $\mathcal{C}$  is  $(s, t)$ -feasible, then  $(\varphi^*, \pi^*)$  is a feasible solution of (7). Note that (7d) is fulfilled (as equality) because  $\sum_{e \in \mathcal{E}} \sum_{v \in \mathcal{V}} \varphi_{ev}^* = 0$  and  $\sum_{e \in \mathcal{E}} \sum_{v \in \mathcal{V}} \pi_{ev}^* = \sum_{e \in \mathcal{E}} \sum_{v \in \mathcal{V}} u_{ev} \pi_{ev}^* = \frac{1}{2}$ . The last equality holds because multipliers  $\pi^*$  are positive only for such pairs  $(e, v)$  that either  $v = s$  and  $e \in \mathcal{F}$ , or  $v = t$  and  $e \in \mathcal{B}$ ; for these pairs  $u_{ev} = 1$ .

**Proposition 3.1.** *If cycle  $\mathcal{C}$  is  $(s, t)$ -valid then objective (7a),  $F_{\mathbf{u}}(\varphi^*)$ , is strictly positive and therefore  $\mathbf{u}$  is infeasible.*

*Proof.* The proof follows from the observation that  $F_{\mathbf{u}}(\varphi^*) = \Delta \cdot (|\mathcal{F}| - |\mathcal{B} \cap \mathcal{I}(s)| + |\mathcal{B}| - |\mathcal{F} \cap \mathcal{I}(t)|)$ . Clearly,  $(\mathcal{B} \setminus \mathcal{I}(s)) \cup (\mathcal{F} \setminus \mathcal{I}(t)) \neq \emptyset$  implies  $|\mathcal{B} \cap \mathcal{I}(s)| + |\mathcal{F} \cap \mathcal{I}(t)| < |\mathcal{F}| + |\mathcal{B}| = |\mathcal{C}|$ , and hence  $F_{\mathbf{u}}(\varphi^*) > 0$ .  $\square$

Finally, observe that  $F_{\mathbf{u}}(\varphi^*) = \frac{1}{2}(1 - \frac{n}{|\mathcal{C}|})$  where  $n = |\mathcal{B} \cap \mathcal{I}(s)| + |\mathcal{F} \cap \mathcal{I}(t)|$ .

In [14] Broström and Holmberg examined the question when valid cycles correspond to vertex solutions of (7). In fact, since their version of the dual problem has no objective (see [10]), its solution space is a polyhedral cone and hence the proper question is when a valid cycle corresponds to an extreme ray of the cone. The main result of [14] states that each valid cycle does represent an extreme ray.

## 4 Multiple cycles

Certainly, directed cycles and valid cycles are not sufficient to describe all non-admissible routing configurations, which occasionally can be more complex (see [15]). Probably, the simplest situation of this type is illustrated in Figure 1. Figure 1a depicts a relevant part of a network. Routing configuration  $\mathbf{u}$  is depicted in Figure 1b and Figure 1c: each of the two figures shows routing towards one of destinations  $v = 1$  and  $v = 2$ , respectively. The following convention is used:  $v$  is the destination node; solid arcs correspond to the links  $e$  with  $u_{ev} = 1$ ; dashed arcs correspond to the links  $e$  with  $u_{ev} = 0$ . The structure of circular flows which proves infeasibility of routing vector  $\mathbf{u}$  consists of two oppositely directed flows (of the same size) shown in the figures; the negative flows are compensated on the inner arcs as shown by the two cycles in Figure 1a. It can be verified that optimal objective  $F_{\mathbf{u}}^*$  is equal to  $\frac{2}{3}$  while the size of each flow is equal to  $\varphi^* = \frac{1}{6}$ ; this is achieved by setting  $\pi_{ev}^*$  to  $\frac{1}{6}$  whenever  $u_{ev} = 1$ , and 0, otherwise.

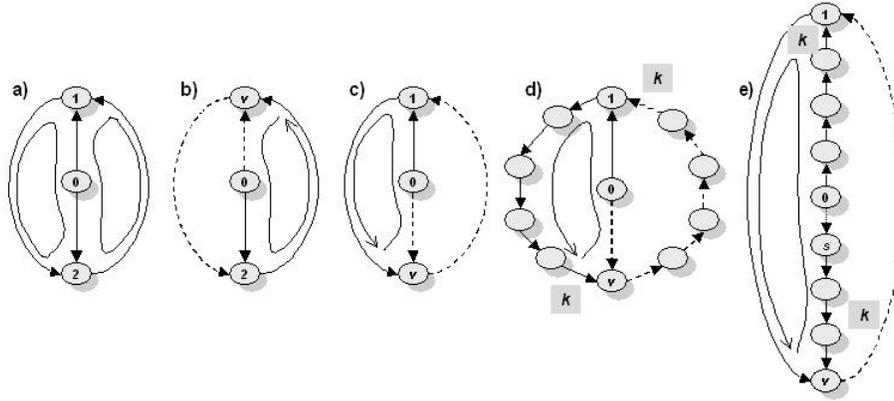


Figure 1: 2-cycle configurations

Notice that there is no obvious relation between the complexity of circular flows and the value of  $F_{\mathbf{u}}^*$  (recall that we always have  $0 \leq F_{\mathbf{u}}^* \leq 1$ ), and that the optimal objective depends on the proportion of links which carry positive flows. If the network from Figure 1a is modified by replacing each outer arc with  $k$  arcs and the routing is changed appropriately (Figure 1d), then  $\varphi^* = \frac{1}{2 \cdot (k+2)}$ ,  $F_{\mathbf{u}}^* = \frac{k+1}{k+2}$ , and  $F_{\mathbf{u}}^* \rightarrow 1$ ,  $k \rightarrow \infty$ . On the other hand, if each inner arc is replaced with  $k$  arcs (Figure 1e) (with all  $u_{ev}$  set to 1 except for the arc from 0 to  $s$ ), these values become  $\varphi^* = \frac{1}{2 \cdot (2k+1)}$ ,  $F_{\mathbf{u}}^* = \frac{2}{2k+1}$ , and  $F_{\mathbf{u}}^* \rightarrow 0$ ,  $k \rightarrow \infty$ . We note that such linear sequences of  $k$  arcs do not mean that the network is odd since it can be a part of a larger network.

A solution revealing routing infeasibility may have to consist of any number of cycles, as illustrated in Figure 2. The structure depicted in Figure 2a assumes outer nodes as destinations. If for each destination  $v$  the configuration of routing is such as in Figure 2b, then the cyclic structure must consist of  $n$  cycles presented in Figure 2c, and  $F_{\mathbf{u}}^* = \frac{2}{3}$ . However, if for each destination node the routing configuration is slightly modified (cf. Figure 2d), other types of cycles are possible; the smallest cycle and the largest cycle are depicted in the figure. Thus, although the  $n$ -cycle structure from Figure 2c still proves non-admissibility of the routing configuration, a simpler 2-cycle structure shown in Figure 2e does the job as well. In this case  $F_{\mathbf{u}}^* = \frac{n}{n+2}$ , which suggests that maximizing  $F_{\mathbf{u}}$  may lead to generating cyclic structures consisting of fewer cycles. We

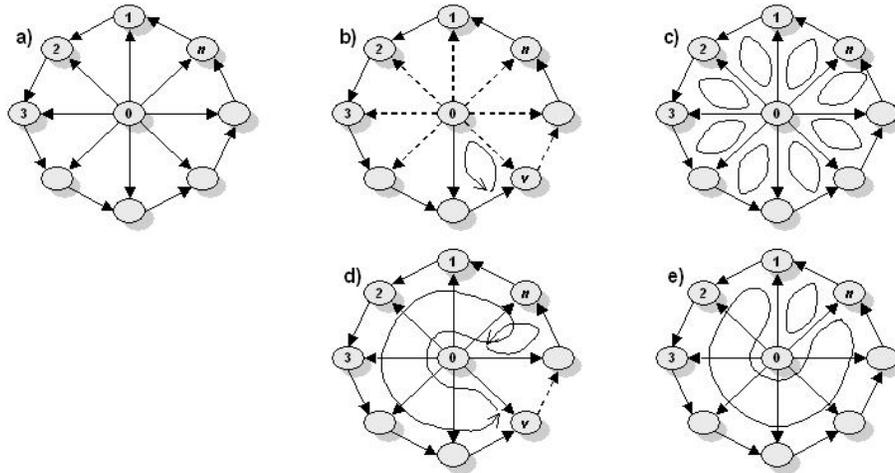


Figure 2:  $n$ -cycle configurations

observe that if in Figure 2a any single outer arc is deleted then the routing vector becomes admissible.

## 5 Final remarks

In this paper we have studied admissibility of binary routing vectors  $\mathbf{u}$  characterizing shortest-path routing with traffic flow split of the OSPF/ECMP type. We have derived an LP problem formulation  $\mathbf{F}(\mathbf{u})$ , depending on  $\mathbf{u}$ , whose optimal objective is equal to 0 if, and only if, vector  $\mathbf{u}$  is admissible as a routing configuration. Further, we have investigated possible forms of the problem solutions corresponding to non-admissible routing configurations. These solutions reveal interesting features of such configurations that can be used in the two-phase approach for solving shortest-path routing problems. The approach, described for example in Section 7.4 of [1], solves a flow allocation problem in phase 1, and then, in phase 2, checks whether the resulting routing vector  $\mathbf{u}^*$  is admissible. If it is not, valid inequalities for variables  $\mathbf{u}$  are added to the problem of phase 1, and the procedure is repeated. A novel way of generating such valid inequalities for phase 1 from solutions of problem  $\mathbf{F}(\mathbf{u}^*)$  is demonstrated in Section 4 of the companion paper [9]. For these inequalities it is generally convenient to have as few links with non-zero flows as possible, so considerations such as in Section 4 are of interest.

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