

# Spectral bounds for the maximum cut problem

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## 1 Introduction

The maximum cut problem this paper deals with can be formulated as follows. Given an undirected simple graph  $G = (V, E)$  where  $V$  and  $E$  stand for the node and edge sets respectively, and given weights assigned to the edges:  $(w_{ij})_{ij \in E}$ , a cut  $\delta(S)$ , with  $S \subseteq V$  is defined as the set of edges in  $E$  with exactly one endnode in  $S$ , i.e.  $\delta(S) = \{ij \in E \mid |S \cap \{i, j\}| = 1\}$ . The weight  $w(S)$  of the cut  $\delta(S)$  is the sum of the weights of the edges it contains:  $w(S) = \sum_{ij \in \delta(S)} w_{ij}$ . The problem is then to find a cut with maximum weight.

This is a very classical problem in combinatorial optimization that is known to be NP-hard [6] in general. However some positive results have appeared in the literature (see e.g. [3] for a survey), namely the fact that it can be solved in polynomial time for some restrictions on the topology and/or the edge-weight function (e.g. [2, 5, 8, 9]). Also in the 90's a major breakthrough was done by Goemans and Williamson [7] who introduced a 0.878-approximation algorithm for the case of nonnegative weights on the edges. Their procedure is based on a semidefinite programming (SDP) formulation of the problem:

$$(SDP) \begin{cases} \max \frac{1}{2} \sum_{1 \leq i < j \leq n} w_{ij} (1 - z_{ij}) \\ z_{ii} = 1, \forall i \in \{1, \dots, n\} \\ Z \succeq 0 \end{cases}$$

where  $Z$  stands for the matrix of order  $n$  with entries  $z_{ij}$ . The formulation (SDP) is known to be equivalent to an eigenvalue optimization problem introduced earlier by Delorme and Poljak [4]. In this paper we shall see how the upper bound provided by these formulations can be improved.

For, consider the following formulation of the maximum cut problem:

$$\mathcal{P}_I \begin{cases} w^* = \max \sum_{1 \leq i < j \leq n} w_{ij} (x_i + x_j - 2x_i x_j) \\ x_i \in \{0, 1\}, \forall i \in \{1, \dots, n\} \end{cases}$$

Relaxing the constraints on integrality and substituting variables  $y_i = 2x_i - 1$  we get the following relaxation:

$$\mathcal{P}_L \begin{cases} Z_{\mathcal{P}_L}^* = \max \frac{1}{2} \sum_{1 \leq i < j \leq n} w_{ij} - \frac{1}{4} y^t W y \\ -1 \leq y_i \leq 1, \forall i \in \{1, \dots, n\} \end{cases}$$

where  $W$  is the symmetric matrix of order  $n$  with entries  $w_{ij}$  and a zero diagonal. In fact we can show the latter gives the optimal objective value of the maximum cut problem.

**Proposition 1.1.** *The following equality holds:  $Z_{\mathcal{P}_L}^* = w^*$ .*

We shall see next how the formulation  $\mathcal{P}_L$  can be used to derive bounds on the optimal objective value  $w^*$ .

The paper is organized as follows. In Section 2 we introduce a lower bound before we present upper bounds in Section 3. Then in Section 4 we discuss on several features of some upper bounds introduced, namely the complexity status of their computation and a way to compute approximations in polynomial time. Some preliminary computational results on small instances are reported in Section 5. Finally, we draw some conclusions and perspectives in Section 6.

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## 2 Lower bound on the optimal objective value

In what follows we shall denote the eigenvalues of the matrix  $W$  with  $\lambda_1 \leq \dots \leq \lambda_n$  and associated eigenvectors with  $v_1, \dots, v_n$ . Also we shall make use of the following classical notations for some norms used:  $\|u\|_2 = \sqrt{\sum_i u_i^2}$ ,  $\|u\|_\infty = \max_i |u_i|$  and  $\|u\|_1 = \sum_i |u_i|$ .

The following lower bound can be easily derived from the formulation  $\mathcal{P}_L$ .

**Proposition 2.1.** *Let  $\lambda_1 \leq \dots \leq \lambda_p < 0$  stand for the negative eigenvalues of the weight matrix  $W$ , then the following holds:*

$$w^* \geq \frac{1}{2} \sum_{1 \leq i < j \leq n} w_{ij} + \max_{i=1, \dots, p} \frac{-\lambda_i}{4\|v_i\|_\infty^2} \geq \frac{1}{2} \sum_{1 \leq i < j \leq n} w_{ij} - \frac{\lambda_1}{4}$$

## 3 Upper bounds on the optimal objective value

From the formulation  $\mathcal{P}_L$  we trivially derive the following upper bound on the optimum, already mentioned by Alon and Sudakov [1]:

$$w^* \leq \frac{1}{2} \sum_{1 \leq i < j \leq n} w_{ij} - \lambda_1 \frac{n}{4}. \quad (1)$$

Note that by definition the weight of any cut does not depend on the diagonal entries of the weight matrix  $W$ , whereas nonzero diagonal entries may change the upper bound (1). From the formulation  $\mathcal{P}_L$  we can derive the following upper bound taking nonzero diagonal entries into account:

$$w^* \leq \frac{1}{2} \sum_{1 \leq i < j \leq n} w_{ij} + \frac{1}{4} \sum_{1 \leq i \leq n} w_{ii} - \lambda_1 \frac{n}{4}. \quad (2)$$

Now considering the diagonal entries of the weight matrix  $W$  as variables one may look for the values leading to the best upper bound, i.e. the ones minimizing the right-hand side in (2).

From now on we shall assume  $W$  stands for the original weight matrix, i.e. with zero diagonal entries, while the modified diagonal entries will be represented by the vector  $u$ . Then the weight matrix with modified diagonal entries is denoted by  $W + \text{Diag}(u)$ , with  $\text{Diag}(u)$  corresponding to a  $n \times n$  matrix with diagonal  $u$  and zero elsewhere. Minimizing the right-hand side in (2) leads to the upper bound:

$$w^* \leq \min_{u \in U} \frac{1}{2} \sum_{1 \leq i < j \leq n} w_{ij} - \frac{n}{4} \lambda_1 (W + \text{diag}(u)) \quad (3)$$

with  $U = \{u \in \mathbb{R}^n \mid \sum_i u_i = 0\}$ .

Bound (3) can still be improved. Let  $d_j$  stand for the Euclidean distance between the set of vectors in  $\{-1, 1\}^n$  and the subspace generated by the eigenvectors  $(v_1, \dots, v_j)$  of the matrix  $W + \text{Diag}(u)$  corresponding to the  $j$  smallest eigenvalues. Then we can prove the following result.

**Proposition 3.1.** *The following inequality holds:*

$$w^* \leq \frac{1}{2} \sum_{1 \leq i < j \leq n} w_{ij} - \lambda_1 \frac{n}{4} - \frac{1}{4} \sum_{1 \leq j \leq n-1} d_j^2 (\lambda_{j+1} - \lambda_j).$$

It can be shown, among others, that the bound of Proposition 3.1 is tight for  $C_5$  (a 5-odd hole) and complete graphs with odd order (whereas the basic SDP formulation is not for both cases). In the next section we shall investigate further the computation of the distances  $(d_j)_{j=1}^{n-1}$  involved in the expression of the last bound given.

## 4 On computing upper bounds: complexity and approximations

Generally computing the exact value of the bound given in Proposition 3.1 is difficult. Indeed we can namely show the following.

**Proposition 4.1.** *Computing  $d_{n-1}$  is NP-hard.*

Obviously the bound of Proposition 3.1 remains valid if one restricts the last sum in the right-hand side to the  $k$  first terms, for some integer  $k \in \{1, \dots, n-1\}$ . If  $k$  is fixed then this restriction of the bound can be computed exactly in polynomial time.

**Proposition 4.2.** *For fixed  $k$ , the distances  $d_1, \dots, d_k$  can be computed in polynomial time.*

Also trivially the bound of Proposition 3.1 remains valid if one replaces the exact values of the distances  $(d_j)_{j=1}^{n-1}$  by nonnegative lower bounds. Let  $V_j$  stand for the subspace generated by the eigenvectors  $(v_1, \dots, v_j)$  corresponding to the smallest eigenvalues of  $W + \text{Diag}(u)$ , and  $E(V_j)$  stand for the  $n \times j$  matrix with the vectors  $v_j$  as columns. Then determining the value  $d_j$  comes down to solving the following program:

$$\begin{cases} \max y^t E(V_j) E^t(V_j) y \\ y \in \{-1, 1\}^n \end{cases} \quad (4)$$

which is difficult in general. Anyway an approximation  $d'_j$  that is also a lower bound for  $d_j$  can be computed in polynomial-time, e.g. by solving (approximatively) a SDP relaxation for the formulation (4). Experiments involving such approximations are under work.

## 5 Preliminary computational experiments

We mentioned in Section 4 that computing all the distances  $d_j$  involved in the expression of the upper bound in Proposition 3.1 is difficult in general. However so as to get some idea of the quality of this bound we considered computing it exactly for some small instances. We report in Table 1 for each instance:

- $Z^*$ : the optimal objective value,
- $U_{SDP}$ : the upper bound from the SDP relaxation (see Section 1),
- $UA$ : the upper bound given by the formula in Proposition 3.1.

Also for each upper bound we report the value of the ratio (upper bound)/ $Z^*$ . Some elements about the instances considered, except the two first which correspond to the well-known Petersen and Coxeter graphs:

- $C_p$ : cycle with length  $p$ ,
- $W_{-p}$ : wheel with  $p-1$  spokes,
- $R(n, m)$ : graph randomly generated with  $n$  vertices and  $m$  edges.

For each instance we considered the cost function with value 1 on each edge (i.e. the *unweighted* version of the max-cut problem). The results point out some improvements of the formula given in Proposition 3.1 over the upper bound provided by the SDP relaxation. Further investigations aiming at better characterizing the quality of the bounds introduced here and some of their approximations are being carried out.

## 6 Perspectives

In this paper we introduced upper and lower bounds for the max-cut problem. All rely on spectral information about the weight matrix (possibly with modified diagonal). Namely upper bounds given here improve on a classical SDP formulation of the problem.

Table 1: Preliminary computational results

Instance	$Z^*$	$U_{SDP}$	ratio $U_{SDP}$	$UA$	ratio $UA$
Petersen	12	12.5	1.042	12.2	1.017
Coxeter	36	37.899	1.053	36.551	1.015
$C_5$	4	4.523	1.131	4	1
$C_7$	6	6.653	1.109	6	1
$C_9$	8	8.729	1.091	8.043	1.005
$C_{11}$	10	10.777	1.078	10.041	1.004
$C_{13}$	12	12.811	1.068	12.078	1.006
$C_{15}$	14	14.836	1.06	14.046	1.003
$C_{17}$	16	16.855	1.053	16.078	1.005
$C_{19}$	18	18.870	1.048	18.051	1.003
$W_{15}$	21	21.875	1.042	21.537	1.026
$W_{16}$	22	23.284	1.058	22.737	1.033
$W_{17}$	24	25	1.042	24.680	1.028
$W_{18}$	25	26.427	1.057	25.575	1.023
$W_{19}$	27	28.125	1.042	27.947	1.035
$W_{20}$	28	29.566	1.056	28.752	1.027
$R(20, 65)$	47	48.947	1.041	47.713	1.015
$R(20, 95)$	62	64.121	1.034	62.857	1.014
$R(20, 128)$	79	81.153	1.027	79.889	1.011

Further work involves evaluating those bounds (or approximations as mentioned in Section 4) experimentally for the max-cut problem. Also, more generally, the approach introduced here could be extended to other combinatorial optimization problems for improving bounds on their optimal objective value.

## References

- [1] Alon, N. and Sudakov, B., "Bipartite subgraphs and the smallest eigenvalue", *Combinatorics, Probability and Computing*, vol.9, pp. 1–12, 2000
- [2] Barahona, F., "The max-cut problem on graphs not contractible to  $K_5$ ", *Operations Research Letters*, vol. 2, pp. 107–111, 1983.
- [3] Ben-Ameur, W., Mahjoub, R. and Neto, J., "Le problème de la coupe maximum". In *Optimisation Combinatoire*, Paschos Eds., Hermès, 2006.
- [4] Delorme, C. and Poljak, S., "Combinatorial properties and the complexity of a max-cut approximation", *European Journal of Combinatorics*, vol. 14, pp. 313–333, 1993.
- [5] Gallucio, A., Loebel, M. and Vondrák, J., "Optimization via enumeration: a new algorithm for the Max-Cut problem", *Mathematical Programming*, vol. 90, pp. 273–290, 2001.
- [6] Garey, M.R. and Johnson, D.S., "Computers and intractability. A guide to the theory of NP-completeness", San Francisco, W.H. Freeman, 1979.
- [7] Goemans, M. and Williamson, D., "Improved Approximation Algorithms for Maximum Cut and Satisfiability Problems Using Semidefinite Programming", *Journal of the ACM*, vol. 42, pp. 1115–1145, 1995.
- [8] Grötschel M. and Nemhauser, G.L., "A polynomial algorithm for the max-cut problem on graphs without long odd cycles", *Mathematical Programming*, vol.29, pp. 28–40, 1984.
- [9] McCormick, S.T., Rao, M.R. and Rinaldi G., "Easy and difficult objective functions for max-cut", *Mathematical Programming*, vol.94, pp. 459–466, 2003.