New inequalities for the p-median Simple Plant Location Problem with Order

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Keywords: Discrete Location - preferences - p-median - Chvátal inequality.

1. Introduction

The p-median Simple Plant Location Problem, p-SPLP, is one of the most broadly studied problems in literature due to its high interest and applicability: [9], [7] or [2] are just some of the many papers devoted to this model. In this problem, non-closest assignments may arise ([8]), that is, a customer may be allocated to an open facility which is not the closest open one to this customer. For example, this happens when the objective function to be minimized is not the total distance but a mix of location and allocations costs.

In other situations, however, customers must be served by the closest open facility. School districting ([3]) or political districting ([6]) are just a couple of examples. Therefore, this requirement must be embedded in the model by means of the *closest assignment constraints* ([4]). Even more, in [5] was shown that the different values are not important, but just their order. Thus, each customer sorts the different potential facilities from the most preferred one up to the most disliked one and, after it has been decided which ones to open, he is allocated to his most preferred open facility. By doing so, we obtain a more realistic model which takes into account the preferences of the customers, preferences due to several reasons: age, educational background, hazards in the route…

The formulation for the p-SPLPO is the same as for the p-SPLP plus the preference constraints. As a consequence, the p-SPLPO is much more difficult solve than the p-SPLP, whose difficulty is well known. The work shown here takes a family of valid inequalities from [1] and strengthens it by means of a Chvátal cut generation technique.

2. The p-SPLPO formulation

Let I be the set of customers and let J be the set of potential facilities, $|I| = m$ and $|J| = n$. The cost of serving the demand of customer i from facility j is c_{ij} and $f_j \geq 0$ is the cost of opening a facility at node j.

Definition 1
Let $i \in I$ and $k, j \in J$. It is said that j is i-better than k *if customer i prefers facility j to facility k. This will be denoted by $j >_i k$. The other usual inequalities are analogously extended into this order of preferences.*

Set $W(i,j)$, $i \in I$, $j \in J$, is the set of facilities strictly i-worse than j, that is,

$$W(i,j) = \{k \in J \,/\, k <_i j\}.$$

If facility j is to be considered in the previous set, then it will be represented by $W'(i,j)$, that is,

$$W'(i,j) = W(i,j) \cup \{j\} = \{k \in J \ : \ k \leq_i j\}.$$

Throughout all the paper preferences will be supposed to be strict, that is, if $k =_i j$ then $k = j$.

*Posthumous work

The model variables are the following:

$$x_{ij} = \text{fraction of the demand of customer } i \text{ which is served from facility } j,$$
$$y_j = \begin{cases} 1 & \text{if facility } j \text{ is open,} \\ 0 & \text{otherwise.} \end{cases}$$

The formulation for the p-SPLPO is:

$$\left\{ \begin{array}{llcrl} \text{Min.} & \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} + \sum_{k \in J} f_k y_k & & & \\ \text{s.t.} & \sum_{j \in J} x_{ij} & = & 1 & \forall i \in I, \\ & x_{ij} & \leq & y_j & \forall i \in I, j \in J, \\ & \sum_{k \in W(i,j)} x_{ik} + y_j & \leq & 1 & \forall i \in I, j \in J, \quad (1) \\ & \sum_{k \in J} y_k & = & p, & \\ & x_{ij} & \geq & 0 & \forall i \in I, j \in J, \\ & y_j & \in & \{0,1\} & \forall j \in J. \end{array} \right.$$

Notice that what we have is the classical p-median SPLP formulation plus the preference constraints (1). As in the SPLP, there is always an optimal solution whose variables x_{ij} take binary values because of the problem being uncapacitated.

3. Generation of the inequalities

Although the following constraint was introduced in [1] for a certain obnoxious location problem, it can be immediately generalized to the context of the p-SPLPO by replacing distances with preferences. We call them T-*inequalities*.

Proposition 2 ([1])
Let $i \in I,\ j \in J$ *and* $T \subseteq W(i,j)$. *If* $|W(i,j)| \geq p$ *and* $|W(i,j) \backslash T| \leq p-1$, *then it holds that*

$$(p - |W(i,j) \backslash T|) \sum_{k \in W(i,j)} x_{ik} \leq \sum_{k \in T} y_k \tag{2}$$

The idea is to combine two specific T-inequalities and then to round the coefficients down to the nearest integer value as in a Chvátal cut procedure.

A quick glance to (2) shows that the left-hand side term depends on both $|W(i,j) \backslash T|$ and set $W(i,j)$ while the right-hand side term depends exclusively on set T.

To illustrate what is going to be done, consider the T-inequalities associated to sets $W(i,j)$ and $W'(i,j)$, $|W(i,j)| \geq p$ and take $T = W(i,j)$ and $T = W'(i,j)$, respectively.

We obtain inequalities

$$p \sum_{k \in W(i,j)} x_{ik} \leq \sum_{k \in W(i,j)} y_k \tag{3}$$

and

$$p \sum_{k \in W(i,j)} x_{ik} + p x_{ij} \leq \sum_{k \in W(i,j)} y_k + y_j. \tag{4}$$

If we add them together, then it holds that

$$p \sum_{k \in W(i,j)} x_{ik} + \frac{p}{2} x_{ij} \leq \sum_{k \in W(i,j)} y_k + \frac{y_j}{2}.$$

Should p be an even number, then the left-hand side term is an integer value. Since $\sum_{k \in W(i,j)} y_k$ is also an integer number and y_j is binary, it holds that

$$p \sum_{k \in W(i,j)} x_{ik} + \frac{p}{2} x_j \leq \sum_{k \in W(i,j)} y_k \tag{5}$$

is a valid inequality tighter than (3).

This idea can be formalized as follows:

Theorem 3 (Chvátal T-inequalities)
Let $i \in I$ and $j_1, j_2 \in J$, with $j_1 \in W(i, j_2)$. If $T \subseteq W(i, j_1)$ is a set such that:

- $|W(i, j_1)| \geq p \geq 2$, *and*
- $|W(i, j_2) \backslash T| \leq p - 1$,

then it holds the following valid Chvátal T-inequality*:*

$$(p - |W(i, j_1) \backslash T|) \sum_{k \in W(i,j_1)} x_{ik} + \varphi^* \sum_{k \in W(i,j_2) \backslash W(i,j_1)} x_{ik} \leq \sum_{k \in T} y_k, \tag{6}$$

where

$$\varphi^* = \begin{cases} \left\lfloor \frac{p - |W(i,j_1) \backslash T|}{|W(i,j_2) \backslash W(i,j_1)|} \right\rfloor & \text{if } \frac{p - |W(i,j_1) \backslash T|}{|W(i,j_2) \backslash W(i,j_1)|} \notin \mathbb{Z}, \\ \frac{p - |W(i,j_2) \backslash T|}{|W(i,j_2) \backslash W(i,j_1)|} & \text{if } \frac{p - |W(i,j_1) \backslash T|}{|W(i,j_2) \backslash W(i,j_1)|} \in \mathbb{Z}. \end{cases}$$

Coefficient φ^* is a strictly positive integer.

If $W(i, j_2) = W'(i, j_1)$, then the new inequality is tighter than the one introduced in Proposition 2:

Corollary 4
Let $i \in I$, $j \in J$ and $p \geq 2$. If $|W(i, j)| \geq p$ and $|W(i, j) \backslash T| \leq p - 2$, then it holds that

$$(p - |W(i, j) \backslash T|) \sum_{k \in W(i,j)} x_{ik} + (p - |W(i, j) \backslash T| - 1) x_j \leq \sum_{k \in T} y_k. \tag{7}$$

Of course, the idea can be extended to more than two facilities. However, each new facility increases highly the analysis.

Theorem 5
Let $i \in I$, $j_1, j_2, j_3 \in J$ and $T \subseteq W(i, j_1)$ be such that:

- $|W(i, j_1)| \geq p$,
- $j_1 \in W(i, j_2)$, $j_2 \in W(i, j_3)$,
- $|W(i, j_2) \backslash W(i, j_1)| \leq p - 1$,

- $|W(i,j_3)\backslash W(i,j_2)| \leq p-1$, *and*
- $|W(i,j_3)\backslash T| \leq p-1$,

then it holds that

$$(p - |W(i,j_1)\backslash T|) \sum_{k \in W(i,j_1)} x_{ik} + \varphi_1^* \sum_{k \in W(i,j_2)\backslash W(i,j_1)} x_{ik} + \\ + \varphi_2^* \sum_{k \in W(i,j_3)\backslash W(i,j_2)} x_{ik} \leq \sum_{k \in T} y_k, \quad (8)$$

where $(\varphi_1^*, \varphi_2^*)$ *is an optimal objective value of the problem*

$$(P_{\varphi_1,\varphi_2}) \left\{ \begin{array}{ll} \textit{Max.} & \{\varphi_1(\beta_1) = \lfloor \beta_1 \rfloor, \varphi_2\} \\ \textit{s.t.} & \beta_1 |W(i,j_2)\backslash W(i,j_1)| + \\ & \varphi_2 |W(i,j_3)\backslash W(i,j_2)| < p - |W(i,j_1)\backslash T|, \\ & 1 \leq \varphi_2 < \beta_1, \\ & \beta_1 \in \mathbb{R}, \varphi_2 \in \mathbb{Z}. \end{array} \right.$$

Particularly, for $W(i,j_3) = W'(i,j_2)$ and $W(i,j_2) = W'(i,j_1)$, the optimal points of problem $(P_{\varphi_1,\varphi_2})$ can be determined easily.

Corollary 6
Let $i \in I$, $j_1, j_2 \in J$ *and* $T \subseteq W(i,j_1)$. *If* $W(i,j_2) = W'(i,j_1)$, $|W(i,j_1)| \geq p$ *and* $|W(i,j_1)\backslash T| \leq p-3$, *then it holds that*

$$(p - |W(i,j_1)\backslash T|) \sum_{k \in W} x_{ik} + (p - |W(i,j_1)\backslash T| - 1 - \lambda)\, x_{ij_1} + \lambda x_{ij_2} \leq \sum_{k \in T} y_k, \quad (9)$$

where $\lambda \in \left\{1, \ldots, \left\lfloor \frac{p - |W(i,j_1)\backslash T| - 1}{2} \right\rfloor \right\}$.

Notice that if it was considered the "extension" $\lambda = 0$, the resulting inequality would be (7), a Chvátal T-inequality with just two facilities.

Acknowledgments

The work of the authors has been funded by Plan Nacional de Investigación Científica, Desarrollo e Innovación Tecnológica (I+D+I), together with ERDF funds, projects TIC2003-05982-C05-03 and MTM2006-14961-C05-04.

In memoriam

This paper is dedicated to our colleague and friend Lázaro Cánovas, who left us recently at the early age of 38.

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