Recognizing matrix bid properties

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1. Introduction

A combinatorial auction is an auction where multiple items are for sale simultaneously to a set of buyers. In a combinatorial auction a buyer is allowed to place bids on subsets of the items. These subsets are sometimes called *bundles*. The auctioneer decides to accept some of the bids and to allocate the items accordingly to the bidders. The main advantage of a combinatorial auction is that it allows a bidder to express his preferences to a greater extent. In a combinatorial auction in its most general form, bidders can bid whatever amount they please on any subset of items in which they are interested. The problem of deciding which bidders should get what items in order to maximize the total winning bid value is called the winner determination problem and is NP-hard (Rothkopf, Pekeç & Harstad 1998). Numerous attempts to cope with this computational complexity can be found in literature. One of them is to impose limitations to what a bidder is allowed to bid on a bundle. The matrix bid auction was developed by Day (2004) and is a way to restrict the preferences of the bidders.

2. The matrix bid auction

The matrix bid auction is a multi-item, single-unit combinatorial auction. This means that for each item that is auctioned, only one unit of this item is available. In the matrix bid auction, each bidder must submit a strict ordering of the items in which he is interested. Furthermore, we assume that for each bidder, the extra value an item adds to a set is determined only by the number of higher ranked items in that set, according to the ranking of that bidder.

The ordering of the items is denoted by r_{ij}, which is item i's position in bidder j's ranking, for each $i \in G$ and $j \in B$. This ordering should be strict in the sense that for each bidder j, $r_{i_1 j} \neq r_{i_2 j}$ for any pair of distinct items i_1 and i_2. For instance, if $r_{ij} = 2$, item i is bidder j's second highest ranked item. Furthermore, each bidder j specifies values b_{ijk}, which correspond to the value the bidder is prepared to pay for item i given that it is the k-th highest ranked item in the set that bidder j is awarded. The b_{ijk} values allow to determine the value bidder j attributes to any set $S \subseteq G$. Indeed, bidder j's bid on a set S is denoted as $b_j(S)$ and can be computed as:

$$b_j(S) = \sum_{i \in S} b_{i,j,k(i,j,S)} \tag{1}$$

where $k(i, j, S)$ is the ranking of item i amongst the items in the set S, according to bidder j's ranking. Notice that equation (1) assumes that no externalities are involved, i.e. a bidder's valuation depends only on the items he wins, and not for instance on the identity of the bidders to whom the other items are allocated. The winner determination problem is, given the bids $b_j(S)$ for each set S and each bidder j, to determine which bidder is to receive which items, such that the total winning bid value is maximized. Notice that we assume that each bidder pays what he bids for the subsets he wins.

It should be clear that the value for index k of item i in bidder j's bid can never be higher than the rank r_{ij}. This allows us to arrange the values b_{ijk} as a lower triangular matrix for each bidder j, where the rows correspond to the items, ordered by decreasing rank and the columns correspond to values for k. Hence the name matrix bid (with order). Notice also that bidder j's ranking r_{ij} does not necessarily reflect a preference order of the items. If an item is highly ranked, this merely means that its added value to a set depends on less items than the added value of a lower ranked item. Furthermore, we make no assumption regarding the b_{ijk} values. Indeed, these values may be negative, e.g. to reflect the disposal cost of an unwanted item. Specifying a sufficiently large negative value can also keep the bidder from winning this item in the first place.

Despite the fact that we adopt a restriction on the preferences a bidder can express, the winner determination problem of the matrix bid auction remains NP-hard (Day 2004). Even more, there exists no polynomial-time approximation scheme for this problem, even when all bidders have the same ranking of the items, unless $P = NP$ (Goossens 2006).

Bids in any combinatorial auction in practice are likely to posses some structure. In literature, we find references of both theoretical structures (see e.g. Rothkopf et al. (1998)) and structures in practice (see e.g. van de Klundert, Kuipers, Spieksma & Winkels (2005)). Capturing and understanding this structure is important, since it allows to develop algorithms that can be more efficient than algorithms for a general combinatorial auction. The matrix bid auction, where the incremental value an item adds to a bid on a set is determined only by the number of higher ranked items in that set, imposes one such structure. Thus, the matrix bid auction offers a way of capturing structure that may be present in combinatorial auctions. Furthermore, matrix bid auctions allow for a faster computation due to the restriction on the preferences that is assumed. Finally, the matrix bid auction also offers a compact way of representing preferences.

3. Recognizing matrix bid properties

In this section, we discuss the relationship between the bid function implied by a matrix bid and economic concepts as free disposal (3..1), complement freeness (3..2), decreasing marginal valuations (3..3), and the gross substitutes property (3..4). In particular, we show how to verify efficiently whether a given matrix bid satisfies each of these properties. Since this section deals only with the bid function of a single bidder, the index j that is used to indicate the bidder will be dropped. In literature, many of the economic concepts discussed in this section are in terms of a valuation function. Although for some auctions (e.g. the VCG auction), it has been shown that it is in the bidder's best interest to bid his true valuation, in general, a bidder's bid and his valuation need not be identical. Indeed, strategic considerations may motivate a bidder to express bids that differ considerably from his valuation. Nevertheless, in this section, we ignore this issue and assume equivalence between the notions bid function and valuation function, which is common in studies on bidding languages (see e.g. Nisan (2005)). We refer to Gul & Stacchetti (1999) and Cramton, Steinberg & Shoham (2005) for the definitions used in this section.

3..1 Free disposal

In microeconomics, it is often assumed that agents prefer more to less. In the context of an auction, this means that a bidder is always willing to receive one or more items for free. Consequently, a seller will never get stuck with unsold items and can therefore dispose of any number of items at no cost. Free disposal is a very common assumption in literature on combinatorial auctions (Nisan 2000) and can be defined as follows.

Definition 1 *A bid function b satisfies the free disposal property if*

$$b(S) \leqslant b(T) \qquad \forall S \subseteq T \subseteq G. \tag{2}$$

Notice that this definition is equivalent with the definition of a monotone non-decreasing function. Alternatively stated, this definition implies that disposing an item from a set cannot increase the value of the resulting set.

In combinatorial auctions where a bidder does not communicate bids on every possible subset of items, but rather only on a limited number of subsets of his liking, allocating all items may be problematic for the auctioneer. In its strictest sense, the lack of free disposal would mean that buyers do not accept anything extra beyond what they bid on. Using this interpretation, even finding a solution to the winner determination problem where all items are allocated is NP-complete (Sandholm, Suri, Gilpin & Levine 2002). However, many other approaches allow the auctioneer to allocate all items, without disposal cost. Nisan (2005) assumes that bids of each bidder satisfy the free disposal property. Moreover, if a bidder did not express a bid on a set S, the auctioneer can construct a new bid $b(S)$ equal to the highest bid over all subsets of S. Obviously, the newly created bids also satisfy the free disposal property. A similar approach is followed by Leyton-Brown, Shoham & Tennenholtz (2000), since they allow the auctioneer to create additional bids with value zero for any subset of items, which can then be combined with any of the bids expressed by the bidders (which also satisfy the free disposal property).

The concept of free disposal is, however, also relevant in a combinatorial auction where bidders do express bids on every possible subset, as in the matrix bid auction. In this case, assuming bid functions that satisfy free disposal guarantees the existence of a total winning bid value maximizing allocation in which all items are awarded to some bidder.

Obviously, not every matrix bid satisfies the free disposal property. The matrix framework indeed allows the bidder to take into account a disposal cost which can vary across the items and may lead to one or more sets with a negative valuation. However, imposing that each entry in the matrix bid is non-negative is not sufficient to attain the free disposal property. This is illustrated by the matrix bid below, as $b(y) = 3 > 2 = b(x, y)$.

$$\begin{array}{r|cc} \text{item } x & 1 & \\ \text{item } y & 3 & 1 \end{array}$$

Verifying free disposal can be done in polynomial time for a given matrix bid, as witnessed by the following theorem.

Theorem 1 *Verifying whether a matrix bid b satisfies the free disposal property can be done in polynomial time by solving a shortest path problem.*

Proof. We will show that solving a shortest path problem on an acyclic graph involving $O(m^3)$ nodes and $O(m^4)$ arcs determines whether a matrix bid b satisfies the free disposal property (2). The graph can be described as follows.

The graph contains a source and a sink, and nodes indexed by (i, s, t). The index i refers to item i and ranges from 1 to m. The index t ranges from 1 to r_i, while s ranges from 0 to t. There are arcs from each node (i, s, t) to $(i', s, t + 1)$ and to $(i', s + 1, t + 1)$, for all items i' ranked lower than i (recall that we consider a single bidder). Furthermore, there is an arc from the source to each node $(i, 0, 1)$ and $(i, 1, 1)$, and there is an arc from each node but the source to the sink. Let the cost on the arc from (i, s, t) to $(i', s, t + 1)$ be equal to $b_{i',t+1}$ and let the cost on the arc from (i, s, t) to $(i', s + 1, t + 1)$ be equal to $b_{i',t+1} - b_{i',s+1}$. Analogously, the arcs from the source to each node $(i, 0, 1)$ and $(i, 1, 1)$ have a cost equal to $b_{i,1}$ and zero respectively. Arcs to the sink have a cost equal to zero. This completes the description of the graph. Notice that this graph is acyclic and contains a number of nodes and arcs that is polynomial in the number of items ($O(m^3)$ and $O(m^4)$ respectively). Figure 1 illustrates this graph; arcs with no indication of their cost have a cost equal to 0.

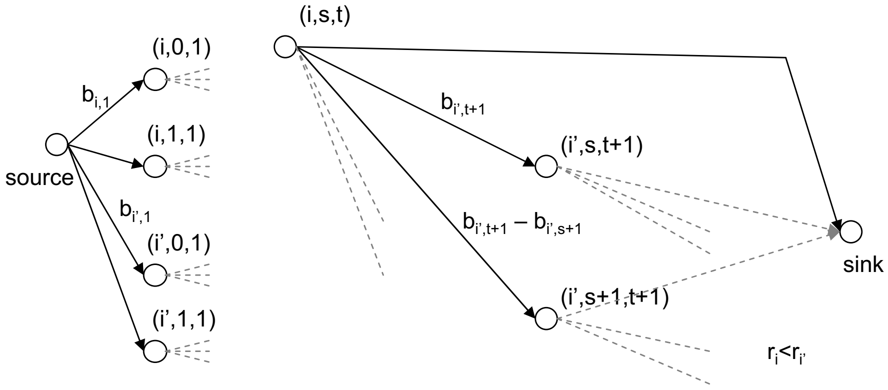

Figure 1: Graph to verify free disposal

The graph described above should be interpreted as follows. Each node (i, s, t) corresponds to a state where s and t items ranked at least as high as item i are present in set S and set $T \supseteq S$ respectively. Selecting an arc from (i, s, t) to $(i', s, t+1)$ therefore corresponds to adding item i' to set T as the $(t+1)$-th best item, but not to S, whereas an arc from (i, s, t) to $(i', s + 1, t + 1)$ corresponds to adding item i' to both set S and set T, as the $(s + 1)$-th and $(t + 1)$-th best item respectively. In this way, each path from source to sink determines what items are to be added to sets S and T, and there is a path from the source to the sink for each possible S and T. Notice that the arcs are such that S will always be a subset of T. We now sketch informally the equivalence between a path in the graph and two subsets of items $S \subseteq T$. We know that in a matrix bid, the value of adding an item i to a set is determined only by the number of higher ranked items. Since the graph contains only arcs from higher ranked items to lower ranked items, the effect of adding an item i to a set on the valuation of this set can be determined, independent of whatever items are added to the set further down the path. This means that the cost of any path from source to sink corresponds to $b(T) - b(S)$, where the items in S and T are determined by the path. Verifying the free disposal property for a given matrix bid can therefore be done by solving a shortest path problem in this graph, which can be done in polynomial time. Using definition 1, a shortest path with non-negative length indicates free disposal. □

3..2 Complement free

Although the difficulty to deal with complementarity or substitution effects in a bidder's valuation in a classic sequential auction is a major motivation for researching combinatorial auctions in the first place, assuming the absence of complementarities is quite common in microeconomic theories. Lehmann, Lehmann & Nisan (2001) state that "in most of microeconomic theory, the consumers are assumed to exhibit diminishing marginal utilities". In their work, they assume that the valuation of a union of disjoint sets is never higher than the sum of the valuations of the individual sets. This notion can be formalized as follows and is also known as subadditivity.

Definition 2 *A bid function b is complement free (or subadditive) if*

$$b(S \cup T) \leqslant b(S) + b(T) \qquad \forall S, T : S \cap T = \varnothing. \tag{3}$$

Theorem 2 *Verifying whether a matrix bid b satisfies the complement free property can be done in polynomial time, by solving a shortest path problem.*

This theorem can be proven by adapting the construction described in theorem 1 (see Goossens (2006)).

3..3 Decreasing marginal valuations

In many practical applications, and also in most of microeconomic theory, it is assumed that the more items an agent has, the less he values an extra item. This concept is called decreasing marginal valuations.

Definition 3 *A bid function b has decreasing marginal valuations if*

$$b(T \cup \{x\}) - b(T) \leqslant b(S \cup \{x\}) - b(S) \qquad \forall S \subseteq T, x \in G. \tag{4}$$

Moulin (1988) showed that a (bid) function has decreasing marginal valuations if and only if it is submodular.

Definition 4 *A bid function b is submodular if*

$$b(S \cup T) + b(S \cap T) \leqslant b(S) + b(T) \qquad \forall S, T \subseteq G. \tag{5}$$

Theorem 3 *Verifying whether a matrix bid b has decreasing marginal valuations can be done in polynomial time, by solving a shortest path problem.*

This theorem can be proven by adapting the construction described in theorem 1 (see Goossens (2006)).

3..4 Gross substitutes property

The gross substitutes property was introduced by Kelso & Crawford (1982) in the context of a labor market, and applied to auctions by e.g. Bevia, Quinzii & Silva (1999) and Bikhchandani & Mamer (1997). The gross substitutes property departs from a price vector p containing prices that are to be paid for each item i. Given a valuation function b, we can define the demand set $D(p)$ of the corresponding bidder given the current price vector p as

$$D(p) = \{\operatorname*{argmax}_{S \subseteq G}(b(S) - \sum_{i \in S} p_i)\}. \tag{6}$$

The gross substitutes property requires that a bidder will continue to demand items for which the price did not rise, if other items have become more expensive. This condition can be defined more formally as follows.

Definition 5 *A bid function b satisfies the gross substitutes property if for all price vectors $p \leqslant q$ (according to a point-wise comparison) and all sets $S \in D(p)$, there exists a set $T \in D(q)$ such that $\{i \in S : p_i = q_i\} \subseteq T$.*

Theorem 4 *Verifying whether a matrix bid b satisfies the gross substitutes property can be done in polynomial time, by solving shortest path problems.*

This theorem can be proven by adapting the construction described in theorem 1 (see Goossens (2006)).

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