Optimising Gradient Constrained Networks with a Single Steiner Point in 3-Space

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1 Introduction

By the *gradient* of an edge pq in 3-space we mean the absolute value of the slope from p to q, which is denoted by $g(pq)$. A *gradient-constrained network* is a network interconnecting a given set of nodes N in 3-space, where each edge in the network is embedded so as to have gradient no greater than a given upper bound m. Such networks have an important application in the underground mining industry, where the network of tunnels must be negotiable by large plant, which limits the gradient of a tunnel to about 1 : 7 [1]. A *gradient-constrained Steiner Minimum Tree (SMT)* S is a gradient-constrained network whose total length is minimum for the given terminal set N. The nodes of the network other than those in N are referred to as *Steiner points*. In the context of underground mining, finding such a network corresponds to minimising the substantial infrastructure costs for tunnels providing access to a given set of ore bodies [1].

S can be regarded as a shortest network in a gradient metric space where the *gradient metric* for edge pq is defined as follows: $|pq|_g = |pq|$ if $g(pq) \leq m$ and $|pq|_g = \sqrt{1+m^2}|p_z - q_z|$ if $g(pq) \geq m$, where $|\cdot|$ denotes the Euclidean length. This is a Banach-Minkowski metric. In gradient metric space every edge pq can be represented as a straight line segment. However, if the gradient of pq is greater than m, then it is straight in the gradient metric space but will be represented as a bent edge prq in Euclidean space, where pr, rq are straight line segments with maximum gradient m. An edge pq is called an *f-edge, m-edge* or *b-edge* if pq is labelled 'f' ($g(pq) < m$), 'm' ($g(pq) = m$) or 'b' ($g(pq) > m$) respectively.

Let a, b, c be three given terminals in a gradient-constrained network. Suppose a degree-3 Steiner point s adjacent to a, b, c has one incident edge cs above s and each of the other two edges as, bs below or on the same horizontal plane as s. Let $\mathcal{L}_{as}, \mathcal{L}_{bs}, \mathcal{L}_{cs}$ denote the respective labels of these edges. Then we say the *labelling* of s is $(\mathcal{L}_{cs}/\mathcal{L}_{bs}\mathcal{L}_{as})$. Degree-4 Steiner points are denoted similarly. In this paper the SMT will have only one Steiner point, and hence there is no confusion in giving the SMT the labelling of the Steiner point. A triangle with labelling $\mathcal{L}_{ab}\mathcal{L}_{bc}\mathcal{L}_{ca}$ is denoted a $\mathcal{L}_{ab}\mathcal{L}_{bc}\mathcal{L}_{ca}$ triangle, and similarly we have a $\mathcal{L}_{ab}\mathcal{L}_{bc}\mathcal{L}_{cd}\mathcal{L}_{da}$ quadrilateral.

Fundamental properties of gradient-constrained Steiner minimum networks are described in [3]. The following will be useful for this paper.

Proposition 1.1 *(1) The degree of a Steiner point s in a gradient-constrained minimum network T is either three or four.*
(2) Up to symmetry there are five (non-degenerate) feasibly optimal labellings (f/ff), (m/ff), (m/mf), (m/mm) and (b/mm) if s is of degree 3 and $m < 1$.
(3) If s is of degree 4 and if $m < 0.38$, then there is only one feasibly optimal labelling (mm/mm). Moreover, two edges, say as and bs, lie in a vertical plane, and so do the other two edges cs, ds.

In general determining the length of an SMT is a difficult global optimisation problem. On the other hand it is much simpler to find the minimum length gradient constrained network that connects the terminals and only the terminals in N. Such a network is known as the *gradient-constrained Minimum Spanning Tree (MST)*, T, and by definition its length is at least that of the SMT. Let L_S be the length of the SMT and L_T the length of the MST, both under the gradient metric. A measure of the improvement, with respect to length, of the SMT over the MST is the *Steiner ratio* L_S/L_T, which is designated ρ.

This paper is concerned with finding the configuration of three terminals that has the minimum Steiner ratio, and the value of the ratio, when edges of both the SMT and the MST are subject to the gradient constraint. It is also concerned with the minimum ratio for configurations of four terminals that have a degree-4 Steiner point.

2 Three Point Configurations

In this section we establish a necessary condition for a three point configuration to have the minimum Steiner ratio, then we partition triangles satisfying the necessary condition according to triangle labelling, and finally we further partition each class of triangle labelling according to SMT labelling. This is to enable the behaviour of ρ to be determined over all suitable triangles.

It is well known that in the Euclidean (unconstrained) case the configuration that gives the minimum Steiner ratio is the equilateral triangle, and here $\rho = \frac{\sqrt{3}}{2}$ ([5]). We define a *g-equilateral triangle* to be a triangle abc in 3-space having $|ab|_g = |bc|_g = |ca|_g$. The following is easy to show using a variational argument, which is demonstrated in [3]:

Theorem 2.1 *A triangle in 3-space having the minimum value of the Steiner ratio is a g-equilateral triangle.*

Due to the gradient metric, which can assign the same length to edges with different Euclidean lengths, there are an infinite number of such triangles. In order to facilitate finding the one(s) with the minimum Steiner ratio we can reduce the size of the set of g-equilateral triangles that need to be considered. It can be shown that by choosing an appropriate orthogonal coordinate system we need only consider triangles in the positive octant, with one of the vertices, say a, at the origin and such that the x-axis lies on the plane of the triangle. We label the other two terminals so that $b_z \leq c_z$. For g-equilateral triangles it is easy to see that $c_x < b_x$. Since scaling does not affect gradients of a triangle or its Steiner ratio, we may assume the g-equilateral $\triangle abc$ has unit side length. It follows that $\rho = \frac{1}{2}L_S$. Note that if $g(ab) = 0$ then $|ac|_g = |bc|_g$ does not give a unique g-equilateral triangle due to symmetry. The ambiguity can be removed by setting $c_x \leq \frac{1}{2}$. We define a *G-equilateral triangle* to be a triangle abc in the positive octant with a at the origin, the x-axis on the plane abc, $b_z \leq c_z$, $c_x \leq \frac{1}{2}$, and $|ab|_g = |bc|_g = |ca|_g = 1$. For the minimum Steiner ratio we need only consider G-equilateral triangles. For a fixed m, two independent variables are needed to describe the vertices a, b, c and all induced functions. Let the inclination of the plane of the triangle abc be β and let $c_x = k$. We define Γ to be the $k - \beta$ parameter domain determined

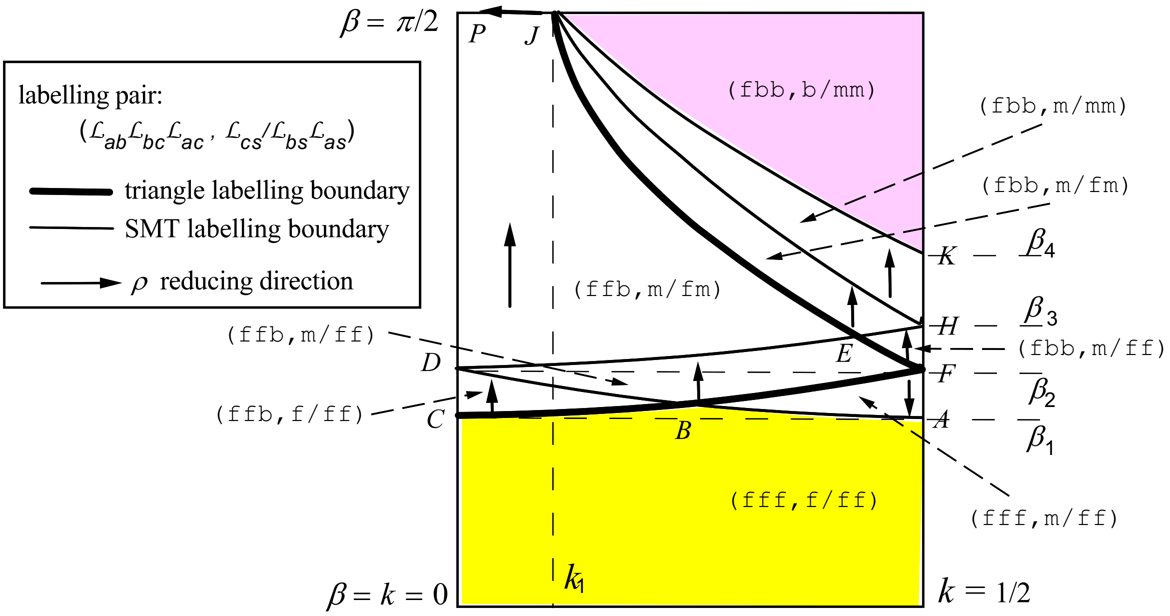

Figure 1: *Partition of* Γ *and direction of reducing* ρ.

by $0 \leq k \leq \frac{1}{2}$ and $0 \leq \beta \leq \frac{\pi}{2}$. It is straightforward to show that there is a bijective mapping of the class of all G-equilateral triangles onto Γ.

There are 27 potential labellings for a triangle, but it can be shown that only 6 can occur in G-equilateral triangles. Detailed proofs of Theorems 2.2 to 2.8 can be found in [6].

Theorem 2.2 *The only feasible labellings of a G-equilateral triangle are fff, ffm, ffb, fmm, fmb and fbb.*

We define the collection of points in Γ that represent G-equilateral triangles having the same labelling as *triangle regions* or simply *regions* of Γ. By Theorem 2.2 Γ has 6 regions, see Figure 1. Because label 'm' is a critical label of 'b' and 'f', Region ffm is a 1-dimensional degenerate region, ie a curve $\mathcal{C}_{CF}$ in Γ, and similarly Region fmb is a curve $\mathcal{C}_{FJ}$. As a result, $\mathcal{C}_{CF}$ separates 2D-Regions fff and ffb, and $\mathcal{C}_{FJ}$ separates 2D-Regions ffb and fbb. By the same reasoning Region fmm degenerates into a point F, which is the intersection of $\mathcal{C}_{CF}$ and $\mathcal{C}_{FJ}$.

Theorem 2.3 $\mathcal{C}_{CF}$ *is given by*

$$g(ac) - m = \beta - \arctan\left(\frac{m}{\sqrt{1-(1+m^2)k^2}}\right) = 0,$$

which is convex and has a positive slope, and $\mathcal{C}_{FJ}$ *is given by*

$$g(bc) - m = \beta - \arctan\left(\frac{m}{\sqrt{1-(1+m^2)(1-k)^2}}\right) = 0,$$

which is convex and has a negative slope.

Remark: G-equilateral triangles with labelling fff, ffm and fmm are Euclidean equilateral triangles; those with labelling ffb and fmb are Euclidean isosceles triangles, the former having no edges horizontal and the latter having edge ab horizontal; those with labelling fbb are irregular, and have edge ab horizontal. In all cases f-edges and m-edges have Euclidean length 1, and b-edges have Euclidean length < 1.

By Proposition 1.1 it can be shown that there are twelve possible labellings of the SMT of a triangle, namely (f/ff), (ff/f), (m/ff), (ff/m), (m/fm) (mf/m), (m/mf), (fm/m), (m/mm), (mm/m), (b/mm) and (mm/b). This can be reduced to five for G-equilateral triangles.

Theorem 2.4 *The only possible labellings of the SMT of a G-equilateral triangle are exactly (f/ff), (m/ff), (m/fm), (m/mm) and (b/mm).*

We define the ordered pair $(\mathcal{L}_{ab}\mathcal{L}_{bc}\mathcal{L}_{ca},\ \mathcal{L}_{cs}/\mathcal{L}_{bs}\mathcal{L}_{as})$ to be a *subregion* of Γ. All points in a subregion of Γ represent G-equilateral triangles that have the same labelling and in addition have SMTs with the same labelling. Theorem 2.5 indicates which ordered pairs are possible and hence defines the subregions.

Theorem 2.5 *(i) The following SMT labellings occur in G-equilateral triangles with labelling fff: (f/ff) and (m/ff).*
(ii) The following SMT labellings occur in G-equilateral triangles with labelling ffb: (f/ff), (m/ff) and (m/fm).
(iii) The following SMT labellings occur in G-equilateral triangles with labelling fbb: (m/ff), (m/fm), (m/mm) and (b/mm).

The proof of Theorem 2.5 is based on properties of labelled Steiner points that are described in [3]. Note that in the gradient constrained problem the Steiner point in general lies above the plane of the triangle, whereas in the Euclidean problem it lies on the plane. Thus the determination of the length of the SMT is more complex in the gradient constrained problem. Each of the triangle regions is partitioned into the admissible subregions bounded by curves representing the critical cases of a Steiner edge labelling changing from 'm' to 'f' or 'b'. These curves are described in the three theorems below and shown in Figure 1.

Theorem 2.6 *The region fff is partitioned by a curve $\mathcal{C}_{AB}$, which is given by*

$$\beta \ - \ \arctan \frac{2m}{\sqrt{3 - m^2 + 2k(\sqrt{3 - 3k^2} - k)(1 + m^2)}} \ = \ 0 \ ,$$

into 2 subregions, namely (fff,f/ff) and (fff,m/ff).

Theorem 2.7 *The Region ffb is partitioned by curves $\mathcal{C}_{BD}$ and $\mathcal{C}_{DE}$ into three subregions, namely (ffb,f/ff), (ffb,m/ff) and (ffb,m/fm).*

Theorem 2.8 *The Region fbb is partitioned by curves $\mathcal{C}_{EH}$, $\mathcal{C}_{HJ}$ and $\mathcal{C}_{JK}$ into four subregions, namely (fbb,m/ff), (fbb,m/fm), (fbb,m/mm) and (fbb,b/mm).*

The equations of curves $\mathcal{C}_{AB}, \mathcal{C}_{BD}, \mathcal{C}_{HJ}$ and $\mathcal{C}_{JK}$ can be explicitly described. Equations of curves $\mathcal{C}_{DE}$ and $\mathcal{C}_{EH}$ cannot be found, though the system of three equations which points on the curves must satisfy can be described. Moreover the existence and uniqueness of the point E on $\mathcal{C}_{FJ}$ has been established. Expressions for the points A, B, C, D, F, H, J and K, which satisfy the necessary triangle and SMT labelling properties, have also been established, see [6].

3 Four Point Configurations with a Degree-4 Steiner Point

We can assume $m < 0.38$, since the maximum gradient constraint of practical interest in the underground mine design is about 0.14. Then by Proposition 1.1(3) a degree-4 Steiner point has labelling (mm/mm), with as, bs lying on one vertical plane and cs, ds on another. By means of the variational argument we can determine which configuration(s) of four points with a degree-4 Steiner point has the minimum Steiner ratio. A *g-equilateral quadrilateral* is defined to be a quadrilateral $abcd$ in 3-space under gradient constraint having $|ab|_g = |bc|_g = |cd|_g = |da|_g$. We label the terminals such that $a_z \leq b_z \leq c_z \leq d_z$.

Theorem 3.1 *The configurations of four terminals in 3-space with a degree-4 Steiner point which have the minimum value of the Steiner ratio are g-equilateral quadrilaterals abcd and abdc. Each of these must have horizontal edges ab and cd.*

There are an infinite number of configurations satisfying the above theorem, since the angle between the vertical planes through ab and cd can have any value from 0 to $\frac{\pi}{2}$. In each, ad, ac, bc, bd are b-edges, since the minimum value of their gradient can be shown to be $\sqrt{2}m$. No partitioning argument is necessary since there is only one subregion, namely (fbfb,mm/mm).

4 Minimum Steiner Ratio

For the 3-point case it is possible to determine the variation of ρ with respect to β in each of the subregions. The details are given in [6]. For Subregions (fff,f/ff) and (fbb,b/mm), shown shaded in Figure 1, ρ is constant at $\frac{\sqrt{3}}{2}$ and $\frac{1}{2} + \frac{1}{4}\sqrt{1+m^2}$ respectively. For Subregion (fff,m/ff), $\frac{d\rho}{d\beta}$ is positive. In each of the other six subregions the derivative is negative. The directions of reducing ρ are illustrated in Figure 1. It follows that the direction of reducing ρ in Subregion (ffb,m/fm) with $\beta = \frac{\pi}{2}$, determines the G-equilateral triangle with minimum ρ. An expression can be found for L_S for G-equilateral triangles represented by points on PJ in Γ, where $\beta = \frac{\pi}{2}$, and differentiating this with respect to k reveals that minimum ρ occurs at $P\ (0, \frac{\pi}{2})$, which represents a G-equilateral triangle with a vertical edge on the vertical plane. For this triangle the Steiner point does lie on the plane of the triangle.

Theorem 4.1 *The minimum Steiner ratio for the whole of the parameter domain* Γ *occurs at* P $((0, \frac{\pi}{2}))$ *and the value is*

$$\frac{1}{2} + \frac{\sqrt{4+3m^2} - 1}{4\sqrt{1+m^2}} \ .$$

We note that the minimum value of ρ approaches $\frac{3}{4}$ as m tends to zero, and indeed for the practical range of m, namely from 0 to 0.14, ρ with the same accuracy is 0.75. This improvement of 25% compares with an improvement of 13% in the unconstrained case. As the gradient metric space is a Banach-Minkowski space, our result agrees with that in [2], namely that for any Banach-Minkowski space the minimum Steiner ratio for three points is $\geq \frac{3}{4}$.

Now consider the four point case. Unlike the 3-point case, every permissible configuration has the same length of SMT and MST and hence the same Steiner ratio, which is given in Theorem 4.2.

Theorem 4.2 *The configuration of four points with a single Steiner point which has the minimum Steiner ratio is one forming a g-equilateral quadrilateral with two horizontal edges. The minimum ratio is* $\frac{2}{3}\sqrt{1+m^2}$.

The minimum degree-4 Steiner ratio tends to $\frac{2}{3}$ as m approaches zero, whether the points are co-planar or in 3-space. This conforms with [2], in which the minimum value of the Steiner ratio for four points in a Banach-Minkowski plane is given as $\frac{2}{3}$. Even for $m = 0.14$, the highest practical value of m, the minimum value of ρ is 0.67, and for $m = 0.38$ it is 0.71. This compares with the minimum value of ρ for four points in Euclidean 3-space, $\frac{1}{\sqrt{3}} + \frac{1}{3\sqrt{2}} = 0.81$, which is achieved with a regular tetrahedron [4]. The general gradient constrained 4-point problem in 3-space, where there could be two Steiner points, has not been solved. From Theorems 4.1 and 4.2 we obtain:

Theorem 4.3 *The configuration of points with a single Steiner point which has the minimum Steiner ratio is one forming a g-equilateral quadrilateral with two horizontal edges. The minimum ratio is* $\frac{2}{3}\sqrt{1+m^2}$.

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