**An integer formulation to the problem of extracting embedded network submatrices**

**R. M. V. Figueiredo**,* *Universidade do Estado do Rio de Janeiro, Instituto de Matemática e Estatística*
**M. Labbé,** *Université Libre de Bruxelles, Département d'Informatique*
**C. C. de Souza,** *Universidade Estadual de Campinas, Instituto de Computação*

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## 1. Introduction

The knowledge of a special structure in a linear programming problem can be used to speed up its resolution. One of these special structures is a network matrix. A matrix $B$ is called a *network matrix* if the elements of $B$ belong to the set {-1,0,+1} and, additionally, every column of $B$ contains at most one element +1 and at most one element -1. Given a row of matrix $B$, the operation of changing the signs of all non-zero row elements is called a *reflection* of this row. Row reflections can create larger network submatrices. A matrix $B$ is called a *reflected network matrix* if there is a set of row reflections that transforme matrix $B$ into a network matrix.

In this work, we consider the problem of detecting a maximum embedded reflected network (DMERN). Given a {-1,0,+1}-matrix $A$, the DMERN problem consists of finding the maximum number of rows that define a submatrix $B$ of $A$ such that $B$ is a reflected network matrix. Let $\nu(A)$ denote this number.

The DMERN problem is known to be a NP-hard problem [2]. A number of heuristics to extract embedded network submatrices have been developed and analyzed (see references in [3, 4]). On the other hand, we do not know any exact approach to solve this problem.

Let $G = (V, E)$ be an undirected graph and $s : E \to \{+, -\}$ be a function that assigns a sign to each edge in $E$. An undirected graph $G$ together with a function $s$ is called a *signed graph*. In the remainder of this text, $G = (V, E, s)$ denotes a signed graph and $G^-$ denotes the subgraph of $G$ with vertex set $V$ and whose edges are the negatives edges in $G$.

In [4], Gulpinar *et al.* show that the DMERN problem is closely related to balancing of subgraphs in signed graphs. The authors prove that the problem of finding a maximum embedded reflected network can be formulated as an optimization problem over the set of all cuts of a graph. More precisely,

$$\nu(A) = \max_{W \subseteq V} \{\alpha((G^W)^-)\}, \tag{1}$$

where $G^W$ is the graph obtained after changing the sign of each edge $(i, j)$ belonging to the cut $\delta(W)$, induced by $W$, and $\alpha(*)$ denotes the size of a maximum independent set of a given graph.

In this work, we propose an integer formulation to the DMERN problem based on the result due to Gulpinar *et al.*.

## 2. The integer programming formulation

Let $G = (V, E, s)$ be a signed graph. Also, let $E^-(E^+)$ denote the set of negative(positive) edges in $E$. For each edge $(i, j) \in E$, we introduce a binary decision variable $w_{ij}$ and, for each vertex $i \in V$, we introduce a binary decision variable $y_i$. Variables $w \in \{0, 1\}^{|E|}$ will be used to define a cut in the graph $G$ and variables

*Corresponding author (rosa@ime.uerj.br).

$y \in \{0,1\}^{|V|}$ will be used to define an independent set in the signed graph $(G^W)^-$, obtained according to variables $w$. The formulation is as follows.

$$
\begin{array}{llll}
\text{Maximize} & \sum_{i \in V} y_i & & \\
\text{subject to} & y_i + y_j \leq 1 + w_{ij}, & \forall\, (i,j) \in E^-, & (2) \\
& y_i + y_j \leq 2 - w_{ij}, & \forall\, (i,j) \in E^+, & (3) \\
& \sum_{(i,j) \in F} w_{ij} - \sum_{(i,j) \in C \setminus F} w_{ij} \leq |F| - 1, & \forall\, \text{cycle}\, C,\ F \subseteq C,\ |F|\ \text{odd}, & (4) \\
& w_{ij} \in \{0,1\}, & \forall\, (i,j) \in E, & (5) \\
& y_i \in \{0,1\}, & \forall\, i \in V. & (6)
\end{array}
$$

The cycle inequalities (4) together with integrality constraints (5) define a cut in the graph $G$ [1]. Consider an edge $(i,j) \in E^-$. In a solution, $w_{ij} = 1$ means that the edge $(i,j)$ belongs to the cut defined by variables $w \in \{0,1\}^{|E|}$. In this case, constraint (2) forbids vertices $i$ and $j$ to be simultaneously in the independent set. Likewise, $w_{ij} = 0$ means that the edge $(i,j) \in E^-$ does not belong to the cut. In this case, the associated constraint in (2) will be redundant and vertices $i$ and $j$ can be simultaneously in the independent set. An similar analysis can be done for constraints (3). Finally, the objective function minimizes the cardinality of the independent set.

A study of the polytope associated with the formulation is presented, including proofs of which constraints of the formulation are facet-defining and the introduction of new classes of valid inequalities. Based on this polyhedral study we develop a branch and cut algorithm to the DMERN problem. Preliminary computational results show the gain from such an approach.

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