# A New Flow Formulation for the Minimum Latency Problem

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**Keywords:** Minimum Latency Problem, Deliveryman Problem, Traveling Salesman Problem

## 1. Introduction

The Minimum Latency Problem (MLP) [7] [1] [14] [5] [3] [17], also called the Traveling Repairman Problem, the Deliveryman Problem and the Traveling Salesman Problem with Cumulative Costs, is a variant of the Traveling Salesman Problem [8] (TSP) in which a repairman is supposed to visit the nodes of a graph in a way to minimize the overall waiting times of the customers located in the nodes of the graph.

The problem was introduced and related to the TSP in 1967, by Conway, Maxwell and Miller [7], when the MLP was known as a type of scheduling problem. For this type of application the MLP can be interpreted as a Single Machine Scheduling Problem, with sequence-dependent processing times, on which the total flow time of the works has to be minimized [9]. According to Goemans and Kleinberg [13], despite the obvious similarities to the classical TSP, the MLP appears to be much less well-behaved from a computational point of view. Remark that in the MLP the objective is to minimize the total latency time of all the customers, while in the standard TSP the objective function is focused on the minimization of the travel time of a single traveling salesman. The MLP is more complex because it incorporates the customers conflicting objectives, instead of working with a single travel time objective, as in the original TSP.

## 2. Related Problems and Algorithms

The related literature will be discussed in this section after a brief illustration about the differences between the Traveling Salesman Problem and the Minimum Latency Problem. For a graph $G$ with six nodes, each edge with a cost $b_{ij}$ we will solve the TSP and the MLP. In the figure (1) we show the graph $G$.

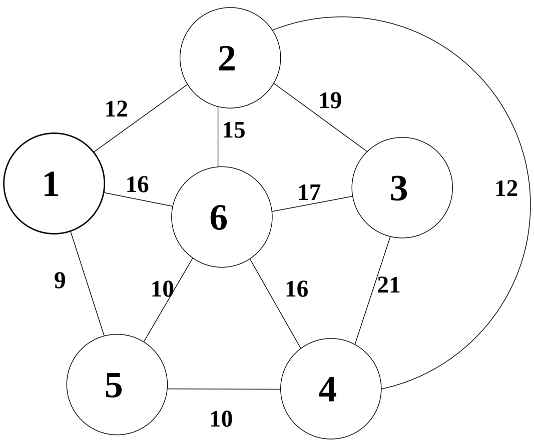


Figure 1: Example.

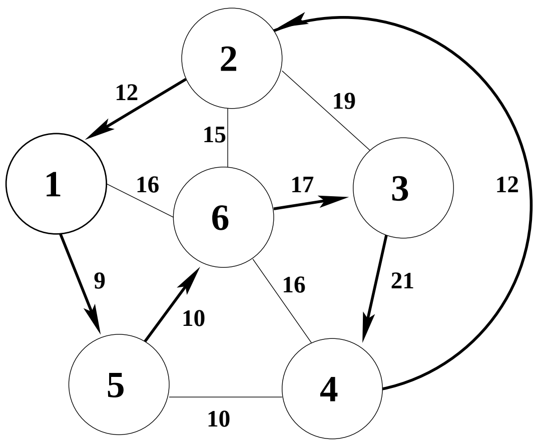


Figure 2: TSP Solution.

The figure (2) shows the TSP solution for this graph. The integer optimal solution for this TSP is $81$ $(9 + 10 + 17 + 21 + 12 + 12)$.

The figure (3) shows the MLP solution for this graph. The integer optimal solution for this MLP is $259$ $(9 +$

$(9+10)+(9+10+12)+(9+10+12+19)+(9+10+12+19+17)+(9+10+12+19+17+16))$ or 259 $(9*6)+(10*5)+(12*4)+(19*3)+(17*2)+(16*1)$.

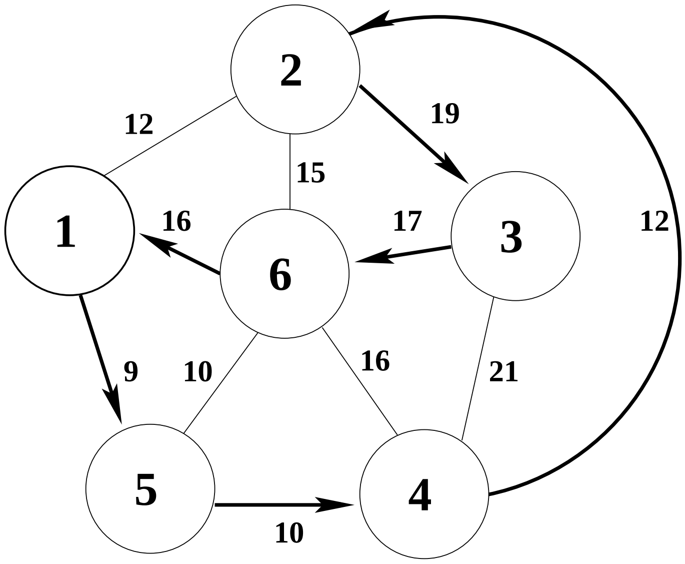


Figure 3: MLP Solution

Both the MLP and the TSP problems are special cases of the Time-Dependent Traveling Salesman Problem (TDTSP) [15] [11]. According to Lucena [14], in the TDTSP, the cost for traversing a link between two cities may vary with the position of this link along the Hamiltonian tour. In the other words, while in the original TSP the cost between two cities is given by $c_{ij}$, in the TDTSP the cost between city $i$ and city $j$ depends on the time period and is given by $c_{ijt}$ for time period $t$. We assume that the time trip between two cities corresponds to one period of time. In the TDTSP the cost function is a metric between the costs of the edge $e$ and the position order of visitation of the $i^{\text{th}}$ nodes, then it incurs a cost $c(e, i)$ . The TSP is a special case where the cost function only depends on the cost of the edge, $c(e, i) = e$. The MLP is the case where the cost is given by $c(e, i) = (n - i) * e$ [5].

The MLP on a metric space is NP-hard and also MAX-SNP-hard [5]. Polinomial time algorithms are only know for very specific graphs, such as paths [1] [12]. See [17] for more details. Due to the NP-hardness, many works have been devoted to the approximation algorithms [5] [13] [3] [2]. The first constant factor approximation for the MLP was $144$ and was given by Blum at al. [5]. Chaudhuri and all [6] improved the ratio to 3.59.

Optimization algorithms for the MLP are provided by Lucena [14], Simchi-Levi and Berman [16] and Fischetti, Laporte and Martelo [10]. The first proposes an enumerative algorithm based on a nonlinear integer programming formulation in which lower bounds are obtained from a Lagrangean relaxation and show exact results up to $30$ nodes. Simcli-Levi and Berman describe a branch-and-bound schema based on a shortest spanning tree relaxation. Fischetti and al. provide a branch-and-bound algorithm based on an integer programming formulation. The paper contais results on the cumulative matroid problem which are used to derive lower bounds. Fischetti, Laporte and Martelo [10] are solved to optimality problems with up to 60 nodes [9]. In 2004, Wu, Hang and Zhan [17] present exact results up to $23$ nodes using dynamic programming with pruning.

A variant problem of the Minimum Latency Problem is the Single Vehicle Delivery Problem (SVDP). This problem was studied by Bianco, Mingozzi, Ricciardelli and Spadoni [4] and characterizes for $1$ vehicle, leaving an origin 1, that covering a Hamiltonian circuit needs to deliver $q_i$ passengers in each node $i$ of the graph $G = (V, E)$. Moreover the driver needs, after to cover all nodes, to deliver $q_1$ passengers in the origin node $1$. In this problem, the visitation order is relevant to know the total cost. Bianco, Mingozzi, Ricciardelli and Spadoni present exact results up to $30$ nodes. The difference between SVDP and MLP, whose makes the SVDP more difficult, is the demand and each node. In SVDP the demand can be different in each node and sometimes is more important to pass first for a place that a bigger number of passengers will be leave, instead of choose a closest place. The SVDP can be model as the MLP just changing the demand in each node in one passenger, that is, the add of heterogeneous demands make the SVDP more difficult to be solved.

## 3. New Formulation for the MLP

In this section we present a new flow formulation for the minimum latency problem. Consider a directed connected graph $G(V, E)$, where $V$ denotes the set of nodes(cities) and $E$ is a collection of arcs (roads). Suppose we have an origin node $o$ and a set of nodes $K$, where $K = V$ and, for each node $k \in K$, a unitary demand should be delivered during a traveling salesman's tour that minimize the overall waiting times of the customers located in the nodes of the graph. In this work we will use the deliveryman problem interpretation. For this problem a mixed linear integer formulation is proposed. We define:

$$x_{ij} = \begin{cases} 1 & \text{if the deliveryman travels across arc}(i, j), \\ 0 & \text{otherwise} \end{cases}$$

$$p_{ij} = \begin{cases} 1 & \text{if the node } j \text{ is the } i^{th} \text{ node visited} \\ 0 & \text{otherwise} \end{cases}$$

$c_{ij}$ = time to cross the arc $(i, j)$

$g_{ij}$: total aggregate flow through arc $(i, j)$.

$$\text{minimize} \sum_{(i,j)\in E} c_{ij} g_{ij} \tag{1}$$

suject to

$$\begin{aligned}
\sum_{(i,j)\in E} x_{ij} &= 1 & \forall j \in V & \quad (2)\\
\sum_{(i,j)\in E} x_{ij} &= 1 & \forall i \in V & \quad (3)\\
\sum_{(o,j)\in E} g_{oj} &= \mid K \mid & & \quad (4)\\
\sum_{(i,k)\in E} g_{ik} - \sum_{(k,j)\in E} g_{kj} &= 1 & \forall k \in K & \quad (5)\\
\sum_{(i,j)\in E} g_{ij} &= \sum_{t=1}^{n} t & & \quad (6)\\
g_{ij} &\leq \mid K \mid x_{ij} & \forall (i,j) \in E & \quad (7)\\
g_{ij} &\geq 0 & \forall (i,j) \in E & \quad (8)\\
x_{ij} &\in \{0,1\} & \forall (i,j) \in E & \quad (9)\\
\sum_{i=1}^{n} p_{ij} &= 1 & \forall j \in V & \quad (10)
\end{aligned}$$

$$\sum_{j=1}^{n} p_{ij} = \quad 1 \quad \forall i \in V \tag{11}$$

$$p_{1o} = \quad 1 \tag{12}$$

$$\sum_{(j,k)\in E} g_{jk} - (\mid K \mid -i+1)p_{ij} \geq \quad 0 \quad \forall i \in V, \forall j \in V \tag{13}$$

$$\sum_{(i,k)\in E} g_{ik} - \sum_{j\in V} p_{ji} = \quad 0 \quad \forall i \in V \tag{14}$$

$$p_{ij} \in \quad \{0,1\} \quad \forall i \in V, \forall j \in V \tag{15}$$

The objective function (1) sums the costs for all the arcs of the network. Equations (2) and (3) are the assignment constraints introduced in 1954 [8] and guarantee that there is just one arc arriving and one arc leaving any node. Constraints (4) says that the total flow leaving the origin is equal to the sum of the demands in all the destination nodes. Constraints (5) assure that the total flow arriving in one node minus the total flow leaving the node is equal to one. The redundant constraint (6) assure that all the flow in the network is equal to $\sum_{t=1}^{n} t$. The constraints (7) state that only can exist flow in a traveled arc, with a total flow is smaller or equal to the sum of the demands in all nodes. The constraints (8) assure that the total flow variables $g_{ij}$ are not negative. The fact that all $x_{ij}$ variables are binary is assured by the constraints (9).

The constraints (10)-(15) are the scheduling constraints and work with the visitation order of the nodes. Constraints (10) - (11) assure that there are just one node been visited in the $i^{th}$ position and vice-verse. Constraint (12) ensure that the first node is the origin. Constraints (13) and (14) are the coupling constraints that link the $p$ and $g$ variables. The fact that all the $p_{ij}$ variables are binary is assure by the constraints (15).

## 4. Computational results

The tests were executed in a Pentium IV with 2.4 GHz and 1 Gbyte of RAM memory. The operational system is Linux. The used solver is Cplex®9.0.

In this work we compare three models. The first model was proposed by Fischetti, Laporte and Martelo [10] and was used as base to the best exact algorithm ever made to solve the MLP. The second model was proposed by Eijl [9] and the third, called Flow, is the model presented in section (3.).

We generate integer costs in the range $[1, 100]$ and triangularizing the resulting matrix through shortest paths. All the instances were asymmetric and complete. For each class and value of $n$, ten instances were generated. The entries in the table give the average and worst values of the running time (in CPU seconds) and the percentage ratio lower bound (GAP LR(%)) that is compute by:

$$GAP\ LR(\%) = 100(\frac{Optimum - LowerBound}{Optimum}) \tag{16}$$

## 5. Conclusion

We have introduced a new flow formulation for a classical problem named Minimum Latency Problem. The Minimum Latency Problem is a heavy variant of the Traveling Salesman Problem and can be defined as a TSP with a different objective function. We show the differences between the TSP and the MLP and also show the related literature of the MLP. We compare our formulation with the formulation of Fischetti, Laporte and Martello [10] and with the formulation of Eijl [9]. In this work we compare the total time and the linear relaxation gap. The three formulations were solved by the software Cplex®9.0.

Table 1: Table of Results (time)

| n | Time (s) Average Fischetti | Time (s) Maximum Fischetti | Time (s) Average Eijl | Time (s) Maximum Eijl | Time (s) Average Flow | Time (s) Maximum Flow |
|---|---|---|---|---|---|---|
| 10 | 22.77 | 37.85 | 11.83 | 51.79 | 0.29 | 0.62 |
| 12 | 2728.33 | 7661.95 | 1150.19 | 3697.63 | 0.52 | 1.1 |
| 15 | **** | **** | **** | **** | 1.56 | 3.65 |
| 20 | **** | **** | **** | **** | 12.21 | 21.74 |
| 25 | **** | **** | **** | **** | 276.78 | 1648.53 |
| 30 | **** | **** | **** | **** | 716.31 | 1332.64 |

Table 2: Table of Results (Gap LR)

| n | Gap LR(%) Average Fischetti | Gap LR(%) Maximum Fischetti | Gap LR(%) Average Eijl | Gap LR(%) Maximum Eijl | Gap LR(%) Average Flow | Gap LR(%) Maximum Flow |
|---|---|---|---|---|---|---|
| 10 | 57.41 | 78.00 | 57.44 | 63.42 | 11.60 | 19.55 |
| 12 | 60.84 | 77.74 | 57.02 | 72.74 | 11.32 | 16.83 |
| 15 | **** | **** | **** | **** | 11.82 | 18.19 |
| 20 | **** | **** | **** | **** | 13.16 | 18.73 |
| 25 | **** | **** | **** | **** | 13.19 | 18.66 |
| 30 | **** | **** | **** | **** | 13.78 | 17.75 |

Through the model (1)-(15) we have solved instances of up to $30$ nodes with a competitive time just using a mixed-integer linear formulation and the software Cplex. Comparing our values with the Fischetti et al. and Eijl formulations, our model is much more fast and have a better linear relaxation gap. With Fischetti et al. and Eijl formulations we just can solve instances up to $12$ nodes and the two models are linear relaxation gaps in the order of 60%. In our model we have solved instances with $30$ nodes faster then Fischetti et al. and Eijl who have solved instances with $12$ nodes. Besides, our linear relaxation gap is much better then the others two models. Besides, we have solved bigger instances then Wu, Hang and Zhan [17]. While they have solved asymmetrical random instances with $20$ nodes in $96.6$ seconds (worse case), see [17], we have solved, in the same Intel Pentium IV 2.4 GHz, random instances with $20$ nodes in $21.74$ seconds, also in the worse case.

We state the importance of the redundant constraints (6) and (10)-(15). Without these constraints our model is almost equal to the Fischetti et al. model [10], but the inclusion of these constraints make our model much more strong. The union of the commodity flow and scheduling interpretation of the MLP is the strength of our work. Besides, the coupling constraints (13)-(14) are tight enough to give good results.

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