Exact two-terminal reliability for the double fan

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1. Introduction

The exact calculation of network reliability in a probabilistic context has long been an issue of practical importance [1, 2, 3, 4]. Even in the case of planar, undirected graphs with perfect nodes, i.e., nodes that never fail, and edges having the same reliability p, it has been shown to be a #P-complete problem [2, 5, 6] (the number of states of the system grows exponentially with the number of edges and nodes). It has therefore induced many approaches: elaboration of bounds based on combinatorics and graph theory, disjoint products and factoring algorithms, Monte Carlo evaluations, studies of the reliability polynomials, etc. However, with the exception of series-parallel reducible graphs, exact results were obtained for very small networks only.

Recently, the two- and all-terminal reliabilities have been given exactly for a few "ladder" architectures, in which arbitrary edge and node reliabilities are taken into account [7, 8, 9]. The obtained analytical expressions reduce to a product of transfer matrices, the size of which depends on the underlying algebraic structure of the graph.

We present here another instance of network architecture that is exactly solvable as regards the two-terminal reliability, which does not vanish as the numbers of edges and nodes go to infinity, and which may not be so easy to solve using either the path-set or the cut-set methods [1]. When all edges have reliability p and all nodes have reliability ρ, we obtain an analytical result for arbitrary large network.

Since the study of the location of zeros of various polynomials connected to graph theory has been — and still is — an important part of the understanding of the underlying properties of graphs, especially in the case of chromatic [10, 11, 12, 13, 14] and Tutte polynomials [15, 16, 17, 18], we shall focus on the interesting structure transitions that can be observed for the location of zeros of the two-terminal reliability polynomial $\mathrm{Rel}_2(p)$ as a function of the node reliability ρ.

2. Analytical expression of the two-terminal reliability

The network architecture under consideration is represented in Fig. 1. We model it as an undirected graph connecting the source S to the destination T through an arbitrary number of n transit sites S_1, ... S_i, ... S_n, which are themselves interconnected by a path, so that the whole graph looks like a double fan. For the sake of simplicity, the node reliabilities are given the same names; we proceed likewise for the edge reliabilities a_n (between S and S_n), b_n (between S_{n-1} and S_n), and c_n (between S_n and T).

Following the general guidelines from our previous work [7, 8, 9], it is possible to show by recursion that the two-terminal reliability $\mathrm{Rel}_2(S \to T)$ is given by a very simple expression, using a product of transfer matrices M_i ($1 \leq i \leq n$)

$$\mathrm{Rel}_2^{(n)}(S \to T) = S\,T\,\widehat{R}_n = S\,T\,(1 - \widehat{S}_n) = S\,T\,\left(1 - (1\,0\,0)\,M_n\,M_{n-1}\,\cdots\,M_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right), \quad (1)$$

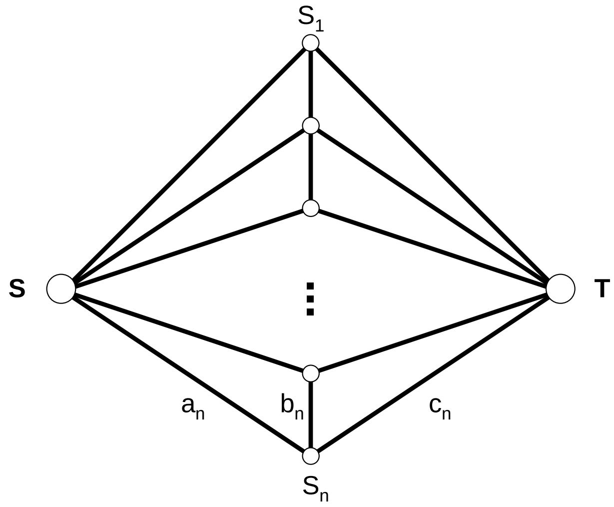

Figure 1: Double fan network, with source S and destination T.

with (for $i = 1$, one should set $b_1 = 0$)

$$M_i = \left(\begin{array}{c|c|c} 1 - a_i\, c_i\, S_i & -a_i\, b_i\, (1 - c_i)\, S_i & -(1 - a_i)\, b_i\, c_i\, S_i \\ \hline (1 - a_i)\, c_i\, S_i & (1 - a_i)\, b_i\, (1 - c_i)\, S_i & -(1 - a_i)\, b_i\, c_i\, S_i \\ \hline a_i\, (1 - c_i)\, S_i & -a_i\, b_i\, (1 - c_i)\, S_i & (1 - a_i)\, b_i\, (1 - c_i)\, S_i \end{array} \right). \tag{2}$$

Note that each edge and node reliability is exactly taken into account, and appears only once in the product, in compliance with the well-known property that reliability is an affine function of each component's reliability.

When edge and node reliabilities are equal to p and ρ, respectively, the characteristic polynomial $\mathcal{P}(x)$ of the *unique* transfer matrix is equal to

$$\mathcal{P}(x) = (x - p\,(1-p)\,\rho)\,\left(x^2 - (1 + p\,\rho - 4\,p^2\,\rho + 2\,p^3\,\rho)\,x + (1-p)\,p\,\rho\,(1 - 2\,p + p^2\,\rho)\right). \tag{3}$$

The eigenvalues of this matrix are $p\,(1-p)\,\rho$ and

$$\lambda_\pm = \frac{1 + p\,\rho\,(1 - 4\,p + 2\,p^2) \pm \sqrt{1 - 2\,p\,\rho\,(1 - 2\,p + 2\,p^2) + p^2\,\rho^2\,(1 - 12\,p + 24\,p^2 - 16\,p^3 + 4\,p^4)}}{2}. \tag{4}$$

The generating function $\mathcal{G}(z) = \sum_{n=0}^{\infty} \mathrm{Rel}_2^{(n)}(S \to T)\, z^n$ can be obtained in a simple way. The existence of $\mathcal{P}(x)$ implies a recursion relation between successive values of $\mathrm{Rel}_2^{(n)}(S \to T)$, and therefore that $\mathcal{G}(z)$ is a rational fraction. Using eqs. (1)–(2) and mathematical software, we find the leading terms of $\mathcal{G}(z)$'s expansion by computing the first values of $\mathrm{Rel}_2^{(n)}(S \to T)$ for $n = 2, 3, 4, \ldots, 10$. The multiplication of this truncated expansion by $(1 - z)\,(z^3\,\mathcal{P}(1/z))$ provides the numerator of $\mathcal{G}(z)$. After a partial fraction expansion, $\mathcal{G}(z)$ reads

$$\mathcal{G}(z) = \frac{\rho^2}{1 - z} - \frac{\rho^2}{2\,\sqrt{\mathcal{B}}} \left(\frac{\mathcal{A} + \sqrt{\mathcal{B}}}{1 - \lambda_+\, z} - \frac{\mathcal{A} - \sqrt{\mathcal{B}}}{1 - \lambda_-\, z} \right), \tag{5}$$

where

$$\mathcal{A} = 1 - p\,\rho\,(1 - 2\,p + 2\,p^2), \tag{6}$$

$$\mathcal{B} = 1 - 2\,p\,\rho\,(1 - 2\,p + 2\,p^2) + p^2\,\rho^2\,(1 - 12\,p + 24\,p^2 - 16\,p^3 + 4\,p^4). \tag{7}$$

It is worth noting that one eigenvalue of the transfer matrix, namely $p\,(1-p)\,\rho$, has utterly vanished from the final result. The two-terminal reliability is then derived from eq. (5)

$$\mathrm{Rel}_2^{(n)}(S \to T) = \rho^2 - \frac{\rho^2}{2\,\sqrt{\mathcal{B}}} \left((\mathcal{A} + \sqrt{\mathcal{B}})\,\lambda_+^n - (\mathcal{A} - \sqrt{\mathcal{B}})\,\lambda_-^n \right). \tag{8}$$

The $n \to \infty$ result is not surprising, since the reliabilities of terminals S and T are always present. The correction to the asymptotic value has a power-law behavior, which is governed by λ_+.

3. Structure transitions of the location of complex zeros for the two-terminal reliability polynomial

An important topic in graph theory has long been the understanding of specific invariants, such as the Tutte and chromatic polynomials. The basic idea is the following: is it possible, from the study of several examples, to infer a general property satisfied by these polynomials for an arbitrary graph ? More specifically, is there some way to characterize these polynomials from the distribution of their zeros in the complex plane?

This approach has been fruitfully adopted by Biggs and collaborators for chromatic polynomials [10, 11, 12]. A few years ago, Brown and Colbourn proposed a conjecture on the location of zeros of the all-terminal reliability polynomials [15]; this conjecture has been shown to be invalid for a few families of graphs [16]. More recently, intense work by statistical physicists and graph theorists on various Potts models (the associated partition function is a particular instance of the Tutte polynomial) and chromatic polynomials [13, 17, 18, 19] has demonstrated that "the study of complex zeros has also become quite popular" [14]. They have shown that these zeros accumulate in the vicinity of sets of algebraic curves or of a few isolated points.

While the all-terminal reliability polynomial is a graph invariant, the two-terminal reliability polynomial is not. Still, the study of its complex zeros may provide some information, especially since we have an extra parameter at our disposal, the node reliability ρ. Our aim is to show that the location of complex zeros exhibits structure transitions at critical values of ρ, which correspond to branchings between different eigenvalues. Because of our analytical solution, we can easily calculate $\mathrm{Rel}_2^{(n)}(S \to T)$ for very large values of n. In the present work, we choose $n = 150$ for which the zeros are very close to their asymptotic limits.

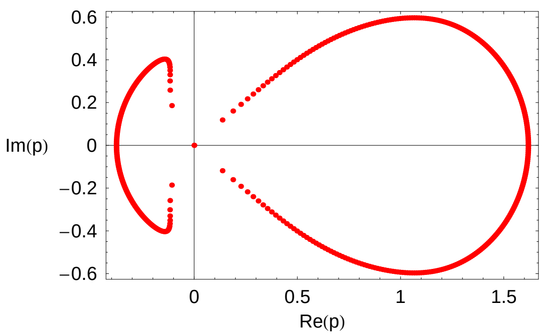

Figure 2: Location of complex zeros of the two-terminal reliability for the double fan, with $n = 150$ perfect transit nodes.

Let us first see what happens for perfect nodes ($\rho = 1$). The location of complex zeros is displayed in Fig. 2. As explained in detail by Beraha, Kahane, and Weiss [20], these zeros aggregate close to sets of curves defined by the fact that two eigenvalues have the same absolute value, which is greater than the absolute values of the remaining eigenvalues. The eigenvalues being here 1, λ_+, and λ_-, we must consider the configurations $|\lambda_\pm| = 1 > |\lambda_\mp|$ and $|\lambda_+| = |\lambda_-| > 1$. For $\rho = 1$, it turns out that the limiting curve verifies $|\lambda_\pm| = 1 > |\lambda_\mp|$. Were n larger, the "sampling" would be better near the origin and we would observe the limiting closed curve, which seems to occur only when the two-terminal reliability tends to a constant value as $n \to \infty$.

What happens when ρ decreases ? The rounded structure of Fig. 2 progressively exhibits angular features, as seen in Fig. 3. When $\rho < 1/2$, a new, antenna-like feature appears. Its origins comes from the fact that $|\lambda_+| = |\lambda_-| > 1$ on a part of the real positive axis. The first critical value is therefore $\rho_{c_1} = \frac{1}{2}$.

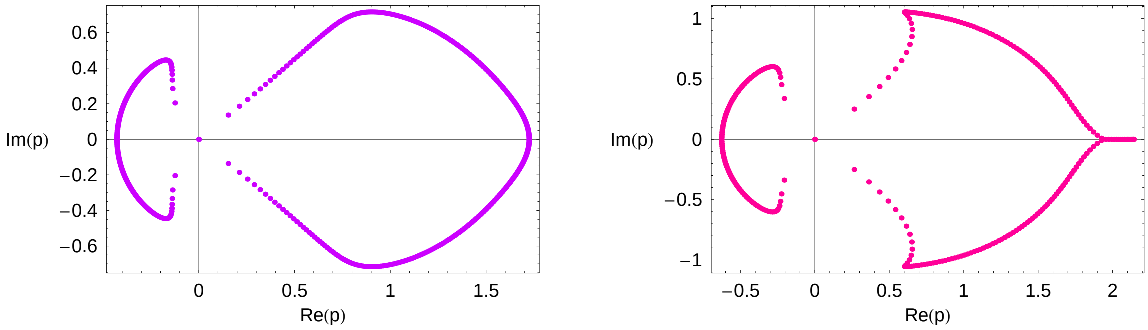

Figure 3: Location of complex zeros for $n = 150$, and $\rho = 0.8$ (left) and $\rho = 0.4$ (right).

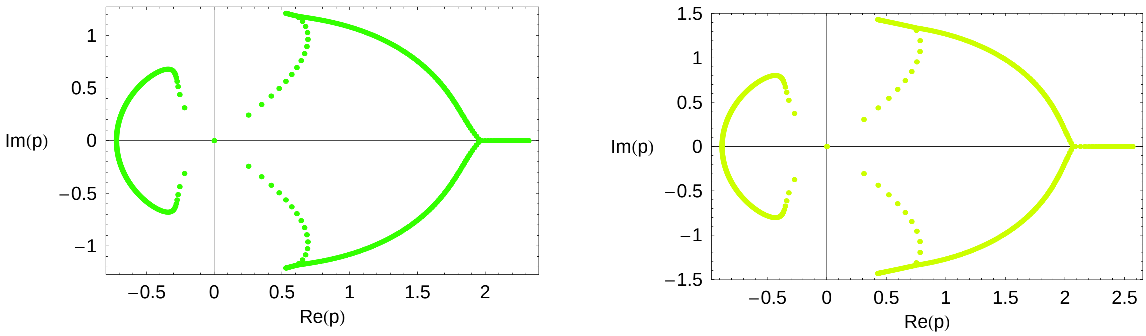

Figure 4: Location of complex zeros for $n = 150$ and $\rho = 0.3$ (left) and $\rho = 0.2$ (right).

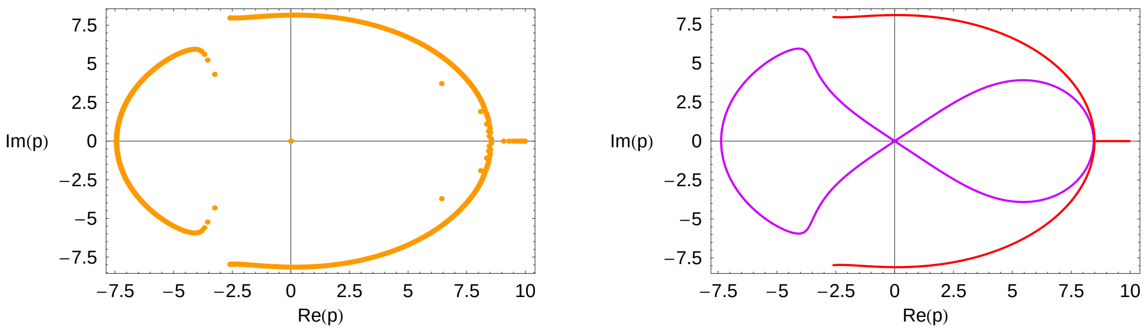

Figure 5: Location of complex zeros for $n = 150$ (left) and their asymptotic limit (right) for $\rho = 10^{-3}$. The red curve is such that $|\lambda_+| = |\lambda_-| > 1$, the purple one corresponds to $|\lambda_+| = 1 > |\lambda_-|$ or $|\lambda_-| = 1 > |\lambda_+|$.

The second critical value is $\rho_{c_2} \approx 0.34598552754$: when $\rho < \rho_{c_2}$, two extra antennae grow from "triple points" (where $|\lambda_+| = |\lambda_-| = 1$), as shown in Fig. 4. Here again, $|\lambda_+| = |\lambda_-| > 1$ along these two antennae. Actually, ρ_{c_2} is one root of the algebraic equation

$$\begin{aligned} 0 \;=\; & 4096 - 686080\,\rho + 6264064\,\rho^2 - 24778112\,\rho^3 + 54394976\,\rho^4 - 69636368\,\rho^5 \\ & + 48091421\,\rho^6 - 9813320\,\rho^7 - 7109996\,\rho^8 + 3208032\,\rho^9 + 426262\,\rho^{10} - 227808\,\rho^{11} \\ & - 25276\,\rho^{12} + 1744\,\rho^{13} + 277\,\rho^{14} + 8\,\rho^{15}\,. \end{aligned} \tag{9}$$

This equation has been obtained by satisfying the constraints $\lambda_+ = \lambda_-$ and $|\lambda_+| = 1$ (with the help of MATHEMATICA for the elimination of variables).

When ρ further decreases, the structure of the complex zeros changes again, as shown in Fig. 5. The third critical value $\rho_{c_3} \approx 0.006982585$ is a solution of

$$0 \quad = \quad -16 + 2336\,\rho - 6428\,\rho^2 + 6148\,\rho^3 - 2193\,\rho^4 + 172\,\rho^5 + 32\,\rho^6. \qquad (10)$$

For $\rho < \rho_{c_3}$, the antennae merge into a single structure, still defined by $|\lambda_+| = |\lambda_-| > 1$. Note that even for $n = 150$, the aggregation of zeros is not uniform for all sets of curves.

Finally, as $\rho \to 0$, the global structure expands from the origin approximately as $(2\,\rho)^{-1/3}$. More precisely, the left-most part of the purple curve is asymptotically given by $-(2\,\rho)^{-1/3} + \frac{7}{12} + O(\rho^{1/3})$, while its right-most part is $+(2\,\rho)^{-1/3} + \frac{1}{2} + O(\rho^{1/3})$. The right-most end of the red curve (the remaining antenna) is $+(2\,\rho)^{-1/3} + \frac{\sqrt{2}}{3}\,(2\,\rho)^{-1/6} + \frac{2}{3} + O(\rho^{1/6})$, while its triple point is found at $+(2\,\rho)^{-1/3} + \frac{5}{9} + O(\rho^{1/3})$. This means that the red and purple curves have an asymptotic, finite separation of $\frac{1}{18}$.

4. Conclusion and outlook

The present results on the two-terminal reliability of the double fan (exact analytical solution, location of the complex zeros of the two-terminal reliability polynomial) may be extended in at least two directions:

- regarding network survivability studies, other recursive families of graphs should be analytically solvable too. They could be useful as building blocks for new and improved sets of bounds/benchmarks for a new generation of algorithms in the general case. These solutions also open the way to sensitivity studies of each component of the network, or can be helpful to network designers taking reliability estimation uncertainty into account [21, 22].
- in a more graph theoretical point of view, we have illustrated the richness of behavior of the structure transitions for the location of complex zeros, and determined critical values of the node reliability ρ. Applied to other architectures, similar studies could provide estimates on the number of eigenvalues, while pure asymptotic reliability gives only access to the eigenvalue of largest modulus.

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