On Time-Dependent Models for Unit Demand Vehicle Routing Problems

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Introduction

The unit-demand Capacitated Vehicle Routing Problem (CVRP) is defined on a given directed graph G = (V,A) with node set V = {1,…,n}, arc set A with an integer weight (cost) c_a associated with each arc a of A as well as a given natural number Q. The problem seeks a minimum cost set of routes originating and terminating at the depot (we assume that node 1 is the depot) with each node in V\{1} visited exactly once and each route containing at most Q nodes (plus the depot). The CVRP is closely related with delivery type problems and appears in a large number of practical situations concerning the distribution of commodities. The book by Toth and Vigo [12] provides surveys on the problem, including variants and solution techniques. Papers by Lysgaard, Letchford and Eglese [10] and Fukasawa et al. [3] discuss the most successful algorithms for solving this problem as well as general demand cases. The paper by Letchford and Salazar-Gonzalez [9] provides a recent comparison of the linear programming relaxation of several formulations presented in the literature and the recent paper by Godinho, Gouveia and Magnanti [5] presents and compares the linear programming relaxation of several so-called multicommodity flow time-dependent formulations.

In this paper we study the relationship between the linear programming relaxation of a well-known single-commodity flow model due to Gavish and Graves [4] (presented in Section 2) and pure time-dependent formulation, presented in Section 3, that is a modified version of the well-known Picard and Queyranne [11] formulation for the TSP (see Section 3). In section 4 we show that the time-dependent formulation implies a new large class of upper bounding and lower bounding flow constraints that are not implied by the linear programming of the single commodity flow model.

2. A Generic Formulation and the Single-Commodity Flow Formulation for the CVRP

Consider the following generic formulation for the CVRP:

$$
\begin{aligned}
\textit{minimize} \quad & \sum_{(i,j)\in A} c_{ij} x_{ij} \\
\textit{subject to} \quad & \sum_{i\in V} x_{ij} = 1 \quad \textit{for all } j \in V \setminus \{1\} \\
& \sum_{j\in V} x_{ij} = 1 \quad \textit{for all } i \in V \setminus \{1\} \\
& \{(i,j) : x_{ij} = 1 \textit{ does not contain routes with more than Q nodes}\} \\
& x_{ij} \in \{0,1\} \textit{ for all } (i,j) \in A.
\end{aligned}
$$

There are several ways to model the implicit route constraints using inequalities involving only the x_{ij} variables (see Letchford and Salazar-Gonzalez [9]). An alternative modelling approach is to use extra variables to express the implicit constraints. Probably, the most well-known such formulation for the CVRP is the following single commodity flow formulation (SCF) due to Gavish and Graves [4] that uses additional flow variables f_{ij} indicating the amount of flow on arc (i,j) (assuming that the depot, node 1, sends one unit of flow to every other node):

$$
\begin{aligned}
\textit{minimize} \quad & \sum_{(i,j)\in A} c_{ij} x_{ij} \\
\textit{subject to} \quad & x_{ij} \in \textit{Assign} \\
& \sum_{i\in V} f_{ij} - \sum_{i\in V\setminus\{1\}} f_{ji} = 1 \quad \textit{for all } j \in V \setminus \{1\} \\
& \sum_{j\in V\setminus\{1\}} f_{1j} = n-1 \\
& f_{ij} \le (Q-1) x_{ij} \qquad \textit{for all } (i,j) \in A,\ i,j \ne 1 \\
& f_{1j} \le Q\, x_{1j} \qquad \textit{for all } j \in V \setminus \{1\} \\
& f_{ij} \ge x_{ij} \qquad \textit{for all } (i,j) \in A,\ j \ne 1 \\
& f_{ij} \ge 0 \qquad \textit{for all } (i,j) \in A.
\end{aligned}
$$

The linking constraints $f_{1j} \le Q\, x_{1j}$ guarantee that the flow on each arc leaving the root does not exceed Q. This restriction, together with the flow conservation constraints and the remaining linking constraints, guarantees that each route cannot contain more than Q client nodes.

3. The Modified Picard and Queyranne Formulation

The well-known Picard and Queyranne formulation [11] for the TSP can be easily modified for the CVRP. As in the original Picard and Queyranne formulation, we use an expanded layered graph. Two copies of node 1, the source and the destination, are on the leftmost position and rightmost position. A node j^h (h=1,…,Q-1), indicates that a copy of node j (j = 2,…,n) is in layer h. The network contains: i) arcs from the source version of node 1 to any node in the levels 1 to Q (we will say that an arc from the source node 1 to any node in the level h, has position h), ii) arcs from nodes in level h to nodes in level h+1, (h = 1,…,Q-1), but we do not allow arcs linking copies of the same original node and, iii) arcs from nodes in level Q to the destination version of node 1.

The main difference between this network and the layered graph construct for the TSP is that now we i) allow arcs leaving the source node 1 to nodes in layers h = 2,…,Q in order to allow vehicle routes with fewer than Q nodes, and ii) allow more than one path linking the source and the destination (to represent several vehicle routes). We obtain the Modified Picard and Queyranne formulation (MPQ) by replacing the implicit part of the generic formulation with the following system

$$
\begin{aligned}
& z_{1j}^{1} = \sum_{i \in V \setminus \{1\}} z_{ji}^{2} && \textit{for all } j = 2,\ldots,n \\
& \sum_{i \in V} z_{ij}^{h} = \sum_{i \in V \setminus \{1\}} z_{ji}^{h+1} && \textit{for all } j = 2,\ldots,n \textit{ and } h = 2,\ldots,Q-1 \\
& \sum_{i \in V} z_{ij}^{Q} = z_{j1}^{Q+1} && \textit{for all } j = 2,\ldots,n \\
& x_{1j} = \sum_{h=1,\ldots,Q} z_{1j}^{h} && \textit{for all } j = 2,\ldots.,n \\
& x_{ij} = \sum_{h=2,..,Q} z_{ij}^{h} && \textit{for all } i, j = 2,\ldots.,n \\
& x_{j1} = z_{j1}^{Q+1} && \textit{for all } j = 2,\ldots,n \\
& z_{ij}^{h} \in \{0,1\} && \textit{for all } (1, j) \in A \textit{ and } h = 1,\ldots.,Q \\
& && \textit{or } (i, j) \in A,\ i, j \neq 1 \textit{ and } h = 2,\ldots,Q \\
& && \textit{or } (i,1) \in A \textit{ and } h = Q+1.
\end{aligned}
$$

In this model, the variable z_{ij}^{h} indicates that arc (i,j) is in a path that contains Q-h+1 nodes after the arc (including node j). The equality constraints imposed upon the z variables simply define a network flow system in this layered graph whose solution (in integer variables) are paths (corresponding to routes in the original graph) from the source version to the destination version of node 1. These constraints permit each path to visit several copies of the same original node. However, in the overall problem, the constraints linking the z_{ij}^{h} with the x_{ij} variables and the assignment constraints in the x variables rule out that situation. Note that the equality constraints relating the x_{ij} and z_{ij}^{h} variables permit us to rewrite the MPQ formulation with only the z variables.

Following Gouveia and Voβ [7] one can show that the linear programming relaxation of the MPQ formulation implies the linear programming relaxation of the SCF formulation. Some computational results (to be presented in the talk) taken from instances with up to 80 nodes will show that, in general, MPQ_L improves on SCF_L. These results suggest that it may be worth knowing what are the inequalities implied by the linear programming relaxation of MPQ but are not redundant in the linear programming relaxation of SCF. Section 4 gives a partial answer to this question.

4. Some Inequalities Implied by MPQ

In this section we describe some inequalities in the space of the variables x_{ij} and f_{ij} that are implied by the linear programming relaxation of MPQ and that are not dominated by the linear programming relaxation of the SCF formulation. Consider the following constraints (whose validity is easy to establish)

$$\begin{aligned} &f_{ij} \le (Q-2)x_{ij} + x_{1i} \quad && \text{for all } (i,j)\in A,\ i,j\neq 1 \\ &f_{ij} \ge 2x_{ij} - x_{j1} && \text{for all } (i,j)\in A,\ j\neq 1. \end{aligned} \qquad (4.1a/b)$$

We can add them to the single commodity flow model to tighten the linear programming relaxation. There are two ways we would like to generalize the constraints (4.1a/b): (i) bound the flow in arcs that are more than 2 arcs away from the depot, and (ii) consider constraints for arc sets instead of a single arc (i,j).

The constraints (4.1a/b) bound the flow in arcs that are at least two arcs away from the depot. To obtain analogous bounds for arcs that are at least 3 arcs away from the depot, for simplicity, we consider only a generalization of the upper bound constraint, by writing it in the form

$$f_{ij} \le (Q-1-k)x_{ij} + Exp(x,k) \qquad \text{for all } (i,j)\in A,\ i,j\neq 1, k=1,\ldots,Q-3$$

with Exp(x,k) denotes a linear term in the *x* variables that depends on k and should equal p ($p \le k$) if the path from node 1 to node i contains k-p+1 arcs. Such constraints can be written by using the concept of jump-sets of arcs (see Dahl [2] and Godinho, Gouveia and Magnanti [5]). Let S_0, S_1,...,S_{k+1} be node-disjoint nonempty sets defining a partition of the node set V with S_0 = {1} and S_{k+1} = {i}. For any pair of node sets A and B let $x(A,B)=\sum_{i\in A,\, j\in B} x_{ij}$. Consider the inequalities

$$f_{ij} \le (Q-1-k)x_{ij} + \sum_{p=0,\ldots,k-1}\ \sum_{q=p+2,\ldots,k+1} (q-p-1)x(S_p,S_q) \quad \text{for all } (i,j)\in A, k=1,\ldots,Q-2$$
$$\text{and all partitions } (S_0,\ldots,S_{k+1}) \text{ of } V \text{ with } S_0=\{1\} \text{ and } S_{k+1}=\{i\}. \qquad (4.2a)$$

Note that when k = 1 we obtain the inequalities (4.1a). Note, also, that if a variable corresponds to an arc "jumping" over t intermediate sets S_j (i.e., (q - p – 1) = t), then its coefficient equals t. It is not difficult to check that the term $\sum_{p=0,\ldots,k-1}\sum_{q=p+2,\ldots,k+1}(q-p-1)x(S_p,S_q)$ appearing in the two inequalities satisfies the conditions previously given for the generic term Exp(x,k).

Next, we analyze constraints of the form

$$\sum_{i\in S',\, j\in S} f_{ij} \le (Q-1-k)\sum_{i\in S',\, j\in S} x_{ij} + ?? \qquad \text{for } S,S'\subset V\setminus\{1\}, S\cap S'=\{\}. \qquad (4.3a)$$

that are stronger than the constraints obtained by adding (4.2a) for all the arcs in the cut [S',S]. For simplicity, we start with the following generalization of (4.1a).

$$\sum_{i\in S',\, j\in S} f_{ij} \le (Q-2)\sum_{i\in S',\, j\in S} x_{ij} + \sum_{i\in S'} x_{1i} \qquad \text{for } S,S'\subset V\setminus\{1\}, S\cap S'=\{\}. \qquad (4.4a)$$

The validity of these constraints is easy to establish. The key item in constraints (4.4a) is the coefficient 1 on the righthand term for every set S. That is, these constraints are stronger than the ones obtained by adding |S|*|S'| corresponding constraints (4.1a). It is not difficult to see how to generalize the more general constraints (4.2a) in a similar way to

$$\sum_{i\in S', j\in S} f_{ij} \le (Q-1-k)\sum_{i\in S', j\in S} x_{ij} + \sum_{p=0,\dots,k\text{-}1}\ \sum_{q=p+2,\dots,k+1} (q-p-1)x(S_p, S_q)$$
$$\textit{for all } S, S' \subset V\setminus\{1\}, S\cap S' = \{\},\ k=1,\dots,Q-2 \qquad \text{(4.5a)}$$
$$\textit{and all partitions } (S_0,\dots,S_{k+1}) \textit{ of } V \textit{ with } S_0=\{1\} \textit{ and } S_{k+1}=S'.$$

Again, we note that the coefficients of the variables within the righthand side summation are independent of S. Thus, for a given k and S, the generalized inequality is stronger than the constraint obtained by adding |S| single node set constraints.

In a similar way we produce the following set of lower bounding inequalities

$$\sum_{i\in S', j\in S} f_{ij} \ge (k+1)\sum_{i\in S', j\in S} x_{ij} - \sum_{p=0,\dots,k\text{-}1}\ \sum_{q=p+2,\dots,k+1} (q-p-1)x(S_p, S_q)$$
$$\textit{for all } S, S' \subset V\setminus\{1\}, S\cap S' = \{\}, k=1,\dots,Q-2 \qquad \text{(4.5b)}$$
$$\textit{and all partitions } (S_0,\dots,S_{k+1}) \textit{ of } V \textit{ with } S_0=S \textit{ and } S_{k+1}=\{1\}.$$

The following result (we omit the proof in this Abstract) shows that a special case of (4.5a/b) are implied by the linear programming relaxation of MPQ. It is still open whether all inequalities (4.5a/b) are implied by the linear programming relaxation of MPQ.

Proposition 4.1 The inequalities (4.5a/b) are implied by the linear programming relaxation of MPQ for all Q if one of the following holds:

i) k = 1
ii) k = 2,…,Q-2 and $S \subseteq S_1 \cup S_2$ (for inequality 4.5a) or k = 2,…,Q-2 and $S' \subseteq S_{k-1} \cup S_k$ (for inequality 4.5b).

We conclude this section by pointing out that inequalities in the space of the variables x_{ij} that are implied by the linear programming relaxation of the MPQ model can be obtained by combining the previous inequalities (and thus, these inequalities are also implied by the linear programming relaxation of the SCF model augmented with (4.5a/b) in the space of the x_{ij} variables). To do this we can follow the procedure used by Gouveia [6] (see also Letchford and Salazar-Gonzalez [9]) for generating the multistar inequalities (see Araque, Hall and Magnanti [1]). Details about new projected inequalities will be given at the presentation.

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