The k edge-disjoint 3-hop-constrained paths polytope

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1 Introduction

Given an undirected graph $G = (V, E)$, two nodes $s, t \in V$ and a positive integer $L \geq 2$, an L-st-path in G is a path between s and t of length at most L. Here the length is the number of edges (also called *hops*) in the path . Given a function $c : E \rightarrow \mathbb{R}$ which associates with every edge $e \in E$ a cost $c(e) \geq 0$, and a positive integer $k \geq 2$, the *k edge-disjoint L-hop-constrained paths problem*(kHPP) consists in finding a minimum cost subgraph of G such that between s and t there exist at least k edge-disjoint L-st-paths. This problem has applications to the design of reliable communication networks.

In this paper, we consider kHPP from a polyhedral point of view. We give an integer programming formulation for the problem and discuss the associated polytope. In particular we show that for $L = 3$, the polytope is given by the trivial, the st-cut and the so-called 3-path-cut inequalities. As a consequence, we obtain a polynomial time cutting plane algorithm for the problem in this case. This generalizes the results of Huygens et al. [3] for $k = 2$ and $L = 3$ and those of Dahl et al. [2] for $L = 2$ and $k \geq 2$.

2 Integer programming formulation

Given a subset $W \subset V$ such that $s \in W$ and $t \in V \setminus W$, the set of edges having one node in W and the other in $V \setminus W$ is called an *st-cut* and denoted by $\delta(W)$. If F is an edge subset which induces a solution of kHPP, it is clear that the incidence vector of F, x^F, satisfies the following inequalities:

$$x(\delta(W)) \geq k, \quad \text{for all } st-\text{cut } \delta(W), \tag{1}$$

$$0 \leq x(e) \leq 1, \quad \text{for all } e \in E. \tag{2}$$

Inequalities (1) are called *st-cut inequalities* and inequalities (2) are called *trivial inequalities.*

In [1] Dahl introduces a class of valid inequalities for the hop-constrained st-path problem as follows. Let $\Pi = (V_0, V_1, \ldots, V_{L+1})$ be a partition of N such that $s \in V_0$, $t \in V_{L+1}$ and $V_i \neq \emptyset$ for all $i = 1, \ldots, L$. Let T be the set of edges $e = uv$ where $u \in V_i$, $v \in V_j$ and $|i - j| > 1$. Then the inequality

$$x(T) \geq 1$$

is valid for the L-path polyhedron. The set T is called an *L-path-cut* (or *L-st-path-cut*). Using the same kind of partitions, these inequalities can be generalized in a straightforward way to the kHPP polytope as

$$x(T) \geq k \tag{3}$$

Constraints of type (3) are called *L-(st-)path-cut inequalities.*

Theorem 2.1 *Let $G = (V, E)$ be an undirected graph and $k \geq 2$. Then kHPP is equivalent to the integer program*

$$\text{Min}\left\{cx; \textit{subject to } (1) - (3),\ x \in \{0,1\}^E\right\}.$$

3 Facets and complete description

In this section we suppose that $L = 3$. Let 3HPP(G) be the polytope associated with 3HPP. First we show when inequalities (1)-(3) define facets

Theorem 3.1 *1) Inequalities $x(e) \leq 1$ define facets of kHPP(G).*

2) An inequality $x(e) \geq 0$ defines a facet of kHPP(G) if and only if $|V| \geq k+3$ or $|V| = k+2$ and e does not belong to either an st-cut or a 3-path-cut of cardinality $k+1$.

Theorem 3.2 *Every st-cut inequality defines a facet of kHPP(G).*

Theorem 3.3 *An inequality (3), induced by a partition $(V_0, ..., V_4)$, with $s \in V_0$ and $t \in V_4$, defines a facet of $P_k(G)$ different from the trivial inequalities if and only if*

1. $|V_0| = |V_4| = 1$;
2. $|[s, V_1]| + |[V_3, t]| + |[s, t]| \geq k + 1$.

The following theorem gives the main result of the paper

Theorem 3.4 *The polytope 3HPP(G) is completely described by the inequalities (1)-(3).*

To prove Theorem 3.4, we consider a directed graph $\tilde{G} = \tilde{V}, \tilde{E}$ which contains the nodes s, t and such that every st-path in $\tilde{G}$ is of length no more than 3 and corresponds to an st-path in G. We consider in $\tilde{G}$ the problem of finding an arc subset of $\tilde{G}$ of minimum weight containg at least k arc-disjoint st-paths. Let kADPP be this problem and kADPP(G) its associated polytope. We show that any optimal solution of kADPP with respect to an appropriate weight system, corresponds to an optimal solution of kHPP in G. We also show that any st-cut inequality and L-st-path-cut inequality, which defines a facet for kHPP(G), corresponds to an st-cut in $\tilde{G}$. Using this we show that inequalities (1)-(3) completely describe 3HPP(G).

As a consequence, we obtain that kHPP can be solved using a polynomial time cutting plane algorithm.

References

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