# A Progressive Hedging Meta-heuristic for Stochastic Network Design

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## 1 Introduction

We focus on the fixed-cost, capacitated, multi-commodity network design (CMND) problem with random demands. We address the stochasticity of demand through discretization of the possibly continuous probability distributions and the generation of a suitable set of scenarios. This makes the NP-Hard CMND problem even more difficult to address due to the significant increase in size brought by the set of scenarios. We propose a heuristic framework inspired by the progressive hedging algorithm of Rockafellar and Wets (1991).

## 2 The Formulation

Let $\mathcal{G} = (\mathcal{N}, \mathcal{A})$ be a network with a node set $\mathcal{N}$ and a directed design arc set $\mathcal{A}$. Let $\mathcal{K}$ denote the set of commodities to be routed using this network, where each commodity $k$ has a single origin $o(k)$, a single destination $s(k)$, and a given demand $w^k$. Let $f_{ij}, c^k_{ij}$ and $u_{ij}$ stand for the fixed cost, the unit routing cost for commodity $k \in \mathcal{K}$ , and the capacity of arc $(i, j) \in \mathcal{A}$, respectively. The arc-based formulation of the deterministic CMND problem can then be written (Magnanti and Wong, 1984):

$$\text{Minimize} \quad z(y,x) = \sum_{(i,j)\in\mathcal{A}} f_{ij} y_{ij} + \sum_{k\in\mathcal{K}} \sum_{(i,j)\in\mathcal{A}} c^k_{ij} x^k_{ij} \tag{1}$$

$$\text{Subject to} \quad \sum_{j\in\mathcal{N}^+(i)} x^k_{ij} - \sum_{j\in\mathcal{N}^-(i)} x^k_{ji} = d^k_i \qquad \forall i \in \mathcal{N}, \forall k \in \mathcal{K} \tag{2}$$

$$\sum_{k\in\mathcal{K}} x^k_{ij} \le u_{ij} y_{ij} \qquad \forall (i,j) \in \mathcal{A} \tag{3}$$

$$y_{ij} \in \{0,1\} \qquad \forall (i,j) \in \mathcal{A} \tag{4}$$

$$x^k_{ij} \ge 0 \qquad \forall (i,j) \in \mathcal{A}, \forall k \in \mathcal{K} \tag{5}$$

where $y_{ij}$ is a binary variable representing whether or not arc $(i, j)$ is selected in the final design, $x^k_{ij}$ is a non-negative continuous variable standing for the flow of commodity $k$ on arc $(i, j)$, and $d^k_i$ equals $w^k$ when $i = o(k)$, $-w^k$ when $i = s(k)$, and 0, otherwise. The objective function (1) accounts for the total system cost, including the fixed costs for the selected arcs and the routing costs for the distribution of commodities. Equations (2) represent the network flow conservation constraints, while relations (3) are the arc linking constraints indicating that the total amount of flow, for all commodities, on an arc should not exceed its capacity if the arc is opened ($y_{ij} = 1$), and must be zero if the arc is closed ($y_{ij} = 0$).

A “natural" formulation of the CMND problem with random demands is as a recourse model, where the selection of the design arcs is performed in the first stage when only the probability distributions of the demands are known. Given the design, the second stage decisions then correspond to the flow distribution for a given realization of demand. The stochastic problem then becomes one of identifying a design that minimizes the fixed cost plus the expected cost of the flow distribution.

Computing the expectation of the distribution costs is generally a daunting task. An often used approach is to approximate the general stochastic CMND problem through a discretization of the probability distributions and the generation of a set of scenarios, each scenario representing a possible realization of the random events. Given a set of scenarios $\mathcal{S}$, the probability $p^s$ of scenario $s$, the demand $w^{ks}$ of commodity $k$ in scenario $s$, and $x_{ij}^{ks}$, the flow amount of commodity $k$ on arc $(i, j)$ if scenario $s$ is realized, the stochastic CMND model minimizes $\sum_{(i,j)\in\mathcal{A}} f_{ij}y_{ij} + \sum_{s\in\mathcal{S}} p^s \sum_{k\in\mathcal{K}} \sum_{(i,j)\in\mathcal{A}} c_{ij}^k x_{ij}^{ks}$ subject to constraints (2) - (5) with the appropriate scenario specification. By modeling uncertainty through scenarios, the stochastic problem becomes a deterministic mixed integer linear program of generally very large dimensions, the number of variables and constraints of the CMND formulation being multiplied by $|\mathcal{S}|$. Decomposition methods are thus of great interest.

## 3 The Decomposition Approach

An equivalent formulation introduces $|\mathcal{S}|$ copies of the first stage decision variables $y$ to obtain $y_{ij}^s$, the design variable for arc $(i, j)$ in scenario $s$, as well as a set of constraints to enforce the requirement that the design in all scenarios must be the same. This so-called *non-anticipatory constraints* are relations (11) in the following formulation:

$$\text{Minimize} \quad \sum_{s\in\mathcal{S}} p^s \left( \sum_{(i,j)\in\mathcal{A}} f_{ij}y_{ij}^s + \sum_{k\in\mathcal{K}} \sum_{(i,j)\in\mathcal{A}} c_{ij}^k x_{ij}^{ks} \right) \tag{6}$$

$$\text{Subject to} \quad \sum_{j\in\mathcal{N}^+(i)} x_{ij}^{ks} - \sum_{j\in\mathcal{N}^-(i)} x_{ji}^{ks} = d_i^{ks} \qquad \forall i \in \mathcal{N}, \forall k \in \mathcal{K}, \forall s \in \mathcal{S} \tag{7}$$

$$\sum_{k\in\mathcal{K}} x_{ij}^{ks} \leq u_{ij} y_{ij}^s \qquad \forall (i,j) \in \mathcal{A}, \forall s \in \mathcal{S} \tag{8}$$

$$y_{ij}^s \in \{0, 1\} \qquad \forall (i,j) \in \mathcal{A} \tag{9}$$

$$x_{ij}^{ks} \geq 0 \qquad \forall (i,j) \in \mathcal{A}, \forall k \in \mathcal{K}, \forall s \in \mathcal{S} \tag{10}$$

$$y_{ij}^s = y_{ij}^t \qquad \forall s, t \in \mathcal{S} \tag{11}$$

Relaxing the non-anticipatory constraints (11) separates the problem according to scenarios. Two main issues must then be addressed: 1) How to address each scenario sub-problem? and 2) How to use globally the local information yielded by the the subproblems, particularly when scenarios do not agree on arc status, to guide the overall search mechanisms toward a unique design vector? With respect to the latter issue, Rockafellar and Wets (1991) proposed the *progressive hedging* algorithm to address continuous stochastic programs. The algorithm separates the problem through an augmented Lagrangean relaxation and converges to a global optimum. At each iteration, an estimation of the solution is computed as the expectation over the current solutions of the scenario subproblems. The latter are then re-solved with adjusted penalties on differences between the global estimation and the local solution.

Convergence cannot be proved for mixed-integer formulations, however (see also Løkketangen and Woodruff, 1996). We therefore use the progressive hedging idea as a meta-heuristic framework. Lagrangean relaxation is also used to separate the problem. Given the boolean nature of the design variables, the scenario subproblems become deterministic CMND problems with a modified objective function. The cycle-based heuristic of Ghamlouche, Crainic, and Gendreau (2003) may then be used to address subproblems. We discuss and compare strategies to approximate the global design and penalize subproblems. These strategies are then embedded into sequential and parallel solution methods, which are described and numerically qualified on an set of problem instances that also provide the means to inquire into the impact of demand correlations on the behavior of the algorithm.

## References


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